Cours Gradins.

Phase ordering: ici paramètre d'ordre scalaire $\phi \in \mathbb{R}$.

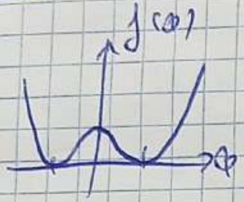
Cas passif ⊕ conservé $\boxed{\partial_t \phi = -\nabla \cdot \vec{j}}$ générique, $j = j[\phi]$.

⇓

existence de F tq "$\vec{j}$ tend à minimiser F"

$$\boxed{F[\phi] = \int dx^d \left(\frac{(\nabla\phi)^2}{2} + f(\phi) \right)}$$

typiquement $f(\phi) = (1-\phi^2)^2$.



two minima : penser aux conditions d'existence de Maxwell : m̂ pente ⊕ m̂ ordonnée à l'origine

$\mu = f'(\phi)$, $P = \phi f'(\phi) - f(\phi)$ pour un mélange passif

⇒ coexistence "Sinclair" : $\begin{cases} \mu_{in} = \mu_{ext} \\ P_{in} = P_{ext} \end{cases}$ (plat.)

$\boxed{SdT - VdP = \sum n_i d\mu_i}$. $\frac{dP}{d\mu} = \phi$.

$\phi_i = \frac{n_i}{V}$.

$\boxed{Ⓐ - \text{Cas simple : non conservé (domaines magnétiques)}}$

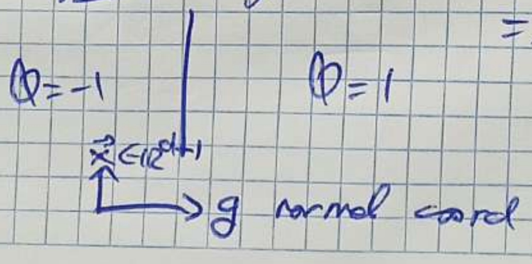
$\partial_t \phi \neq -\nabla \cdot \vec{j}$, plutôt $\boxed{\partial_t \phi = -\frac{\delta F}{\delta \phi}}$.

pense à dynamique de **Glauber**! descente de Gradient.

⇒ ici $\partial_t \phi = \Delta\phi - f'(\phi)$.

pour un mur **flat**: flat ≡ no dep. on $\vec{x} \in \mathbb{R}^{d-1}$.

⇒ $\Delta\phi = f'(\phi)$.



$\Delta\Phi = \frac{\partial^2\Phi}{\partial g^2} + \Delta_x\Phi$. Par sym $\Delta_x\Phi = 0$ ici,

~~$\frac{\partial^2\Phi}{\partial g^2} = f'(\Phi) \rightarrow \underbrace{\frac{\partial^2\Phi}{\partial g^2}\frac{\partial\Phi}{\partial g}} = f'(\Phi)\frac{\partial\Phi}{\partial g}$~~

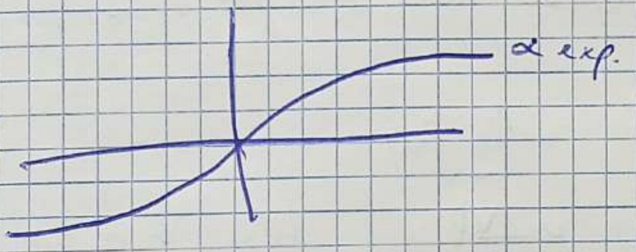
$$\Phi''\Phi' = f'(\Phi)\Phi' = \frac{df}{dg} = \frac{1}{2}\frac{d}{dg}(\Phi'^2)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{d}{dg}(\Phi'^2)dg = 2\int\frac{df}{dg} = 2(f(\Phi_{out}) - f(\Phi_{in})).$$

$$\Rightarrow (\Phi'^2) = 2f(\Phi), \quad \Phi' = \sqrt{2f(\Phi)}.$$

$$\Rightarrow \int_{-\infty}^{\infty}dg\left(\frac{d\Phi}{dg}\right)^2 \underset{\substack{\uparrow \\ \Phi=\Phi(g) \\ d\Phi=\Phi' dg}}{=} \int_{\Phi_{in}}^{\Phi_{out}}\Phi' d\Phi = \sqrt{2}\int_{\Phi_{in}}^{\Phi_{out}}\sqrt{f(\Phi)}\,d\Phi$$

$\boxed{\equiv \gamma}$ tension de surface.



surface tension $\Rightarrow$ energy $E = \gamma A$.

if force F to increase radius by R, by unit area,

$$\underbrace{4\pi F R^2 dR}_{\text{force }(F\times 4\pi R^2)dR} = dE = d(4\pi R^2\gamma) = 8\pi R\gamma dR.$$

$$\Rightarrow \boxed{F = \frac{2\gamma}{R}}$$

et pour $\underline{0}$ accélération, si η = frottement, $\boxed{\eta\frac{dR}{dt} = \frac{-2\gamma}{R}}$.

$$\boxed{\frac{dR^2}{dt} = \frac{-4\gamma}{\eta}} \Rightarrow \boxed{R(t) \sim \sqrt{R(0) - \frac{4\gamma}{\eta}}}$$

collapse of a domain wall! for $\underline{\text{nonconserved}}$ order param

$\propto t^{1/2}$.

temps $\propto R(0)^2$ pour evaporer. "RW".

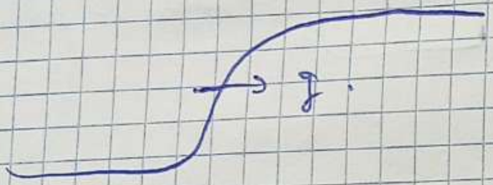
* $\underline{\text{Conserved fields}}$: $\vec{j} = -\nabla\mu$, $\mu = \frac{\delta F}{\delta\phi} = f'(\phi) - \nabla^2\phi$

~~$\nabla^2\phi$ curved~~

$$\nabla\cdot(\vec{\nabla}\phi)^{\frac{\partial\phi}{\partial g}\vec{g}} = \frac{\partial^2\phi}{\partial g^2} + \frac{\partial\phi}{\partial g}\,\nabla\cdot\vec{g} = \nabla^2\phi.$$

$$\Rightarrow \boxed{\mu = f'(\phi) - \partial_g^2\phi - \partial_g\phi\,\underset{K \text{ curvature}}{\underbrace{\nabla\cdot\vec{g}}}\quad(E)}$$

* Maxwell laws for flat: $P_{in} = P_{out}$, $\mu_{in} = \mu_{out}$.



$$\Rightarrow \int \underbrace{\frac{\partial\phi}{\partial g}}_{\text{sharp!}} \times (E)\, dg = \mu\,\overset{\phi_{in}\ \phi_{out}}{\widetilde{\Delta\phi}} \qquad \text{b/c } \mu \text{ is continuous}$$

$$= (\Delta f)(\phi) - K\int(\partial_g\phi)^2$$

$$\Rightarrow \boxed{\mu\,\delta\phi = (\delta f) - K\gamma}.\quad \text{Gibbs-Thompson.}$$

if $\delta f = 0$, $\delta\phi = 2$ (up to normalization)

then $\boxed{\mu = -\frac{K\gamma}{2}.}$ $\underline{\text{GT}}$.

then, movement of the interface

$v \Delta\phi = j_{out} - j_{in}$; ~~assume~~ assume $j_{in} = 0$, (bubble = relaxed).

then $j_{out} = -\frac{\partial\mu}{\partial g}$, so $v = \frac{dR}{dt} = -\frac{1}{\Delta\phi}\frac{\partial\mu}{\partial g}$

if $\Delta\mu = 0$ in the outside, and ~~$\mu = \frac{\Phi}{R}$~~

~~$\mu = \begin{cases} \mu_\infty - \frac{\gamma}{R}, & r \leq R \text{ (no singularity † no current)} \\ \mu_\infty - \frac{\gamma}{r} & r \geq R \end{cases}$~~

~~$\Rightarrow \frac{dR}{dt} = -$~~

$\Delta\mu = 0$, $\mu(+\infty) = \mu_\infty$, $\mu(r \geq R) = \frac{A}{r} + B = \frac{A}{r} + \mu_\infty$.

$\frac{A}{R} + \mu_\infty = -\frac{\gamma}{R} \Rightarrow A = R\mu_\infty - \gamma$.

$(\nabla\mu)_R = -\frac{A}{R^2} = -\frac{\mu_\infty}{R} + \frac{\gamma}{R^2}$

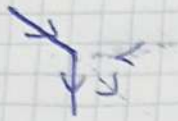
$$\Rightarrow \boxed{\frac{dR}{dt} = \frac{1}{\Delta\phi}\left[\frac{\mu_\infty}{R} - \frac{\gamma}{R^2}\right]}$$ Ostwald ripening.

$R_c =$

† maintenant AMB+.

On part encore de $\partial_t \phi = -\nabla \cdot j[\phi]$: décrire un champ scalaire actif, conservé, pas d'autres degrés de liberté !

ex : particules RTP en interaction coeur dur



etc.

mais pas de nématiques actifs : possèdent un autre deg liberté interne ⇒ le tenseur nématique Q.

~~Lit~~ Littérature : (Cates, Nardini, Solon, ...)

$$\begin{cases} J = -\nabla\mu + \zeta(\nabla^2\phi)\nabla\phi \\ \mu = \frac{\delta F}{\delta\phi} + \lambda|\nabla\phi|^2 \end{cases}$$

expansion de la fonctionnelle non-locale $J[\phi]$ en gradients.

Note : seuls, que les termes à l'éq.. à garder en tête.

ici : λ et ζ cassent sym renversement du temps!

en effet $\left.\begin{matrix}(\nabla^2\phi)\nabla\phi \\ (\nabla\phi)^2\end{matrix}\right)$ even, $(\nabla\mu)_{eq}$ odd.

$\zeta = 0$ ⇒ does not change much.. just quantitative features but no qualitative behavior.

First prediction : both sides of a flat wall have the same chemical pot μ, as in equilib.

However : $\Delta P = \bar{\lambda}\int \phi'^3(x)\,dx$, $\bar{\lambda} = \lambda - \frac{\zeta}{2}$.

① measure difference. How? ⇒ we will see it is due to polar activity at the interface!

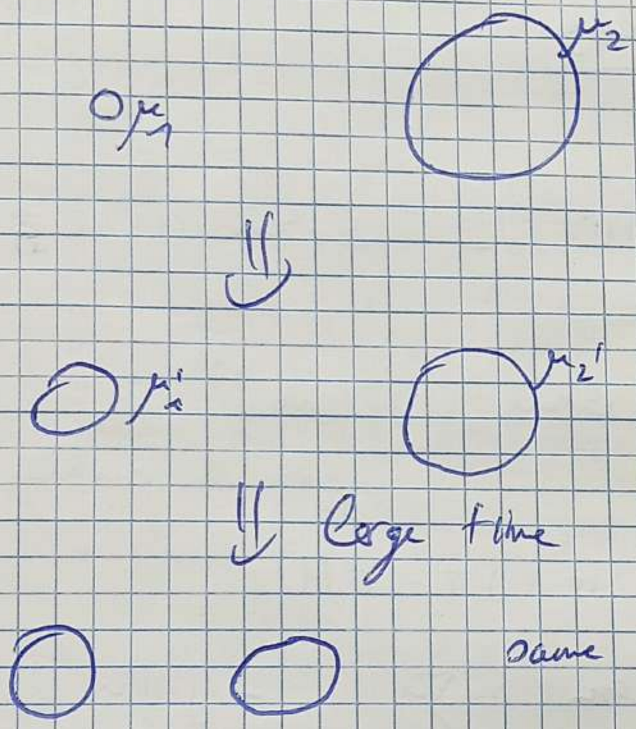
("dry" system exchanges momentum w/ a substrate)

Another comes: change of surface tension. They find an awful expr. for γ_{neg}; importantly it can be $\underline{\underline{<0}}$!!

$$\dot{R} \propto \gamma_{neg}\left(\frac{1}{R_s} - \frac{1}{R}\right), \quad R_s = \frac{(d-1)\gamma}{\varepsilon\Delta\phi\, f''(\phi_1)}$$

Associated G-T: $d\mu = \frac{\gamma(d-1)}{R\Delta\phi}$ for $\gamma<0$,

reverse effect: small droplets have a lower boundary chemical pot than large droplets.



$\mu_1 < \mu_2$
$\Downarrow$
J from 2 to 1 !!

1 has grown – less effect!

same radius!!

Note that $\varepsilon<0$ in this case (chemical pot produced by others.)

Striking: Selection of a large, finite-time radius!

Looks at biological membraneless condensates.

→ huge success.

Can be extended to take into acc conservation of momentum (active model H).

But: big pb why can we decide a gradients]?

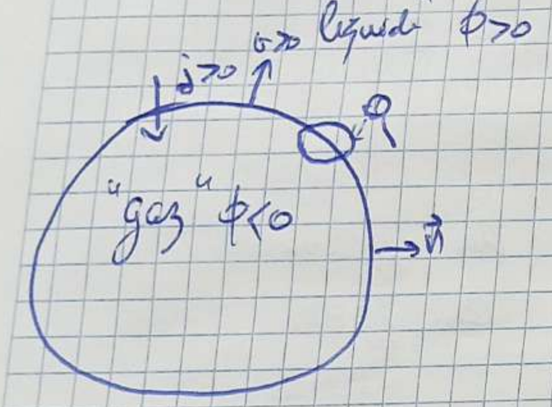
.. we see adding a term gives completely new behavior,

does adding more (~ same order near interfaces) change things?

eg. Burcharić et al go to $O(\nabla^4)$.

- Can we have a systematic, controlled way of dealing w/ activity and large gradients near interfaces??

✱ Active Onsager theory.



trois flux, courbure $H = \nabla\cdot\vec{n}$.

production locale d'entropie ??

1) travail pression: "vP" $= v(\delta\pi - \overbrace{\partial H}^{\text{correction courbure}})$

2) transport de matière, dans le sens $-\nabla\mu$: $-j\,\partial\mu$

3) ⊕ additional energy @ interface

$\Omega\Delta G$.

$\cdot\,\Omega$ = rate of active process

ΔG = ~~free~~ energy production

think of chemical reactions that give heat, or

hard-core particles slow / fast collision.

Now Onsager ⒽΔ:

$$\dot{S} = {}^{t}X\,O\,X,$$ where O is symmetric, and

$$X = \begin{pmatrix} \delta\pi - \gamma H \\ -\delta\mu \\ \Delta G \end{pmatrix}$$ is the vector of fluxes.

$\Rightarrow$

weak curvature ⊕ profiles of P, μ are equal at ~~...~~ $+\ \Delta G = 0$

$$\begin{cases} j = -\lambda\delta\mu + k(\delta\pi - \gamma H) + (\bar{\xi} + \xi H)\Delta G \\ v = \chi(\delta\pi - \gamma H) - k\,\delta\mu + (\vartheta + \vartheta H)\Delta G \\ \Omega = (\cdots) \end{cases}$$

$$O = \begin{pmatrix} \lambda & k \\ k & \chi \\ \cdots & \cdots \end{pmatrix}$$

where $\left[\lambda > 0,\ \chi > 0,\ \text{and},\ \lambda\chi - k^2 > 0.\right]$

(!!) on a complètement coarse-grained $J = J[\phi]$, non-local, compliqué etc... avec juste de la thermo!

$\rightarrow$ on a des relations simples à l'interface, et dans le bulk car le ϕ est slowly varying:

$$J[\phi] \underset{\text{bulk}}{\simeq} M\nabla\mu.$$ $\mu \hat{=}$ effective chem. pot.

L'idée c'est que $j, v \sim \frac{1}{R^2}$, donc pas besoin de tout résoudre. Juste on utilise chgt ref pour écrire

~~$\mathcal{A} = v\phi_{out} + j_{out} \cdot j_{bulk} = M\nabla\mu|_{interface}$~~

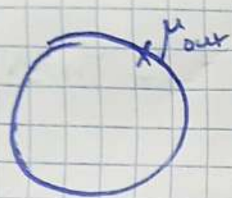
~~$= \mathcal{O}(1/R^2)$.~~

Single droplet: $j_{in} = 0$. then $\begin{cases} j_{out} = j - v\phi_{out} \\ j_{in} = j - v\Phi_{in} \end{cases}$

$$j_{out} = v\Delta\Phi = -M\nabla\mu.$$ ✓ easy

$$v = -\frac{M\nabla\mu}{\Delta\Phi} = \frac{-M}{\Delta\Phi}\nabla($$

$\boxed{\mu_\infty \equiv \varepsilon + \mu_{ss}}$



$$\mu(r>R) = \mu_{ss} + \varepsilon + \frac{R}{r}\left(\mu_{out} - \mu_{ss} - \varepsilon\right) \quad (3D)$$

$$(\nabla\mu)_{out} = \frac{-1}{R}\left(d\mu_{out} - \varepsilon\right) \; ; \; d\mu_{out} = \mu_{out} - \mu_{ss} = \frac{2\gamma}{R\Delta\Phi}$$

$$= \frac{1}{R}\left(\frac{2\gamma}{R\Delta\Phi} - \varepsilon\right).$$

free!

But what is $\underline{\gamma}$?

We see here that $j, v = O(1/R^2)$ generically

($O(1/R) = \varepsilon$ self. consistently).

So, to compute γ, we only need to do it w/ $j = v = 0$.
(or the response of μ
to curvature at leading order)

We write the coexistence densities

$$\phi_{in}(R) = \Phi_{in} + \frac{\eta_{in}}{R} + O(1/R^2)$$

$$\phi_{out}(R) = \Phi_{out} + \frac{\eta_{out}}{R}$$

$H = 1/R$.

then the conditions are $\begin{cases} \delta\mu = \Delta G(\bar{a} + aH) \\ \delta\pi = \Delta G(\bar{b} + bH) + 2\gamma H \end{cases}$

now, use $\mu = f'(\phi)$, $P = \phi f'(\phi) - f(\phi)$

$$\begin{cases} \mu(\phi_{out}(R)) = \mu(\Phi_{out}) + \frac{\eta_{out}}{R} \underbrace{\mu'(\Phi_{out})}_{f''(\Phi_{out})} + \dots \\ P(\phi_{out}(R)) = P(\Phi_{out}) + \frac{\eta_{out}}{R} \underbrace{P'(\Phi_{out})}_{\Phi_{out} f''(\Phi_{out})} + \sim \end{cases}$$

same for P_{in}...

$$\Downarrow \begin{cases} \delta\mu = \overset{\bar{a}}{\overbrace{(\delta\mu)_0}} + (\delta\mu)_1 = a\Delta G \\ \delta P - (\delta P)_0 = (\delta P)_1 = 2\gamma + b\Delta G . \end{cases}$$

solve $\Rightarrow$

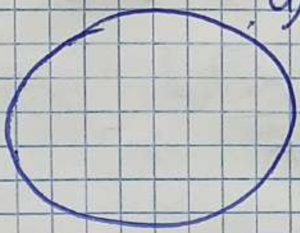
$$\begin{cases} \phi_{out/in}(R) = \Phi_{out/in} + \frac{2\gamma_{act}^{in/out}}{R} + \dots \\ \boxed{\gamma_{act}^{in} = \gamma + \frac{\Delta G}{\lambda K - \zeta^2}\left(\zeta k \Phi_{in} - \lambda \right. \\ \left. + \zeta(k - K\Phi_{in}) \right)} \end{cases}$$

$\Rightarrow$ we can again have $\gamma_{act}^{in} < 0$ ...

~~Capillary waves:~~

- Bubble - in - droplet :

Key idea :

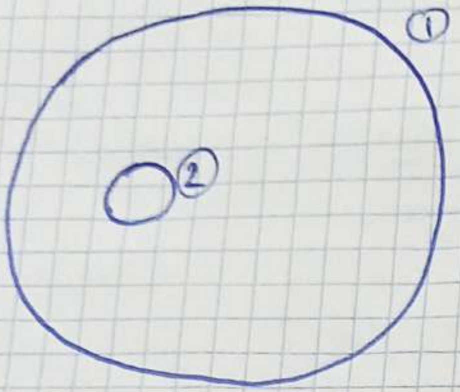


$d\mu = \mu_{?} - \mu_{ss} = \frac{2\gamma}{R\Delta\Phi} + O(1/R^2)$

uniform no Δ, Φ ...

Ext μ_{SD}.

$R_1 \gg R_2 \gg l$.



again $\begin{cases} j^{(1)}_{out} = \Phi^{(1)}_{out} v^{(1)} + j^{(1)} \\ j^{(1)}_{in} = \Phi^{(1)}_{in} v^{(1)} + j^{(1)} \end{cases}$

Hyp: $j^{(2)}_{in} = 0$. (ok in practice.)

$$\Rightarrow \quad v_2 = \frac{-M\nabla\mu|_2}{\Delta\Phi}$$

~~$\Phi^{(1)}_{in} j^{(1)}_{out} - \Phi^{(1)}_{out} j^{(1)}_{in} = j^{(1)} \Delta\Phi^{(1)}$~~

$$v_1 \Delta\Phi_1 = \underbrace{j^{(1)}_{out}}_{-M(\nabla\mu)_{R_1^+}} - \overbrace{j^{(1)}_{in}}^{-M(\nabla\mu)_{R_1^-}}.$$

$-M(\nabla\mu)_{R_1^-}$ → subtle contrib!

$-M(\nabla\mu)_{R_1^+}$ ⇒ single-droplet flux

⇒ suffices to obtain f such that

$$\nabla^2 f = 0, \quad f = 1 \text{ on } ②$$
$$= 0 \text{ on } ①.$$

how to compute this? in the case of off-centered structure?

then $\mu_{between} = \mu_2^{out} f + (\mu_1^{in}$ ~~[illegible]~~$)(1-f)$

$$= \underbrace{(\mu_2^{out} - \mu_1^{in})}_{\Rightarrow d\mu^{out} - d\mu^{in}, \text{ same SS.}} f + \mu_1.$$

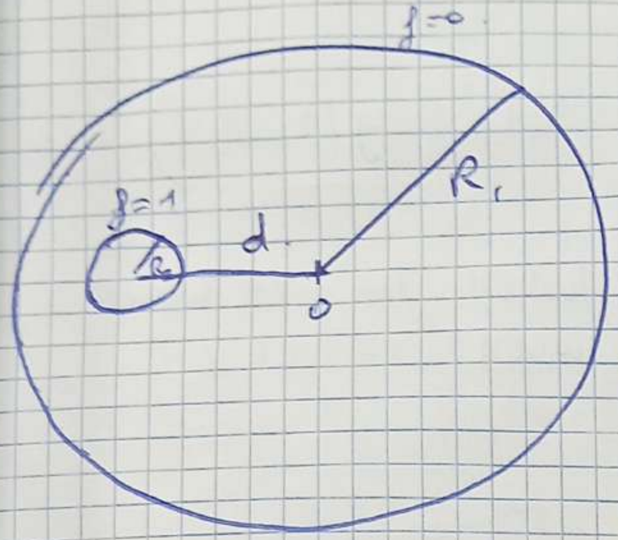
→ can compute all $(\nabla\mu)|_{between}$.

Cases: $d > R_1 + R_2$ (exterior)

$d < R_1 - R_2$ (nested.)

in 3D: iterative charge images (Maxwell)

in 2D: simpler conformal mapping!



Idea: map to concentric sys using a Moebius transform ⇒ conformal

⇓

harmonic (Max

Let $\omega(z) = \dfrac{z-\lambda_+}{z-\lambda_-}$.

We want $\left\{\begin{array}{l} C(0,R_2) \\ C(d,R_1) \end{array}\right. \rightarrow \left\{\begin{array}{l} C(0,\rho_2) \\ C(0,\rho_1). \end{array}\right.$ Choose $\left[\lambda_+ = \dfrac{R_2^2}{\lambda_-}\right]$

then $$\left|\omega(z)\right|_{z \in C(0,R_2)} = \left|\frac{R_2 e^{i\theta} - \lambda_+}{R_2 e^{i\theta} - R_2^2/\lambda_+}\right|$$

$$= \left|\frac{\lambda_+}{R_2}\right| \left|\frac{R_2 e^{i\theta} - \lambda_+}{\lambda_+ e^{i\theta} - R_2}\right| = \left|\frac{\lambda_+}{R_2}\right| = \rho_2.$$

$$= \sqrt{\left|\frac{\lambda_+}{\lambda_-}\right|}.$$

Same for $z \in C(d, R_1)$:

$$\left|\frac{z-\lambda_+}{z-\lambda_-}\right| = \sqrt{\left|\frac{\lambda_+ - d}{\lambda_- - d}\right|} = \rho_1$$

We need to solve for λ_+: let

$$\Delta = \sqrt{(d^2 - (R_1+R_2)^2)(d^2 - (R_1-R_2)^2)} \ \underline{\underline{\in \mathbb{R}^+}}.$$

$$\Rightarrow \boxed{\lambda_\pm = \frac{d^2 - (R_1^2 - \rho_2^2) \pm \Delta}{2d}}.$$

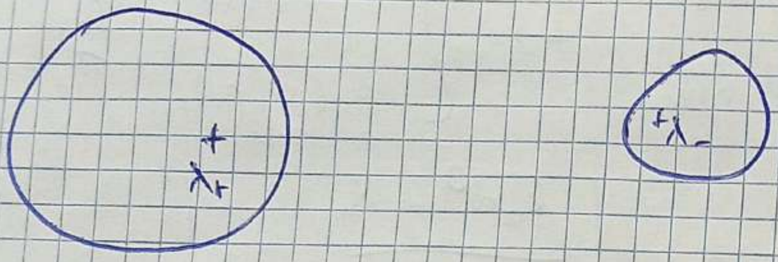
then $\mathrm{Log}\,|\omega(z)|$ is harmonic !

the choose "charge".

$$f(z) = \boxed{\frac{1}{L}} \left(\mathrm{Log}\,|z-\lambda_+| - \mathrm{Log}\,|z-\lambda_-| \right) + cte.$$

$$L = \frac{1}{2} \mathrm{Log} \left| \frac{\lambda_+(\lambda_- - d)}{\lambda_-(\lambda_+ - d)} \right| = \begin{array}{ll} \mathrm{Log}\,|\omega(z)| & \text{for } z \in C(0, R_2) \\ -\mathrm{Log}(|\omega(z)|) & z \in C(d, R_1) \end{array}$$

e.g exterior:



now to obtain the avg speed, we use

$$\vec{U} = \frac{\boxed{d}}{S_d} \overset{\text{Paul.}}{\iint} \underset{\text{obtained from } \nabla f.}{\boxed{\vec{v}\cdot\vec{e_z}}}\, dS_d.$$

how to compute $\vec{D} = \frac{1}{\boxed{S_d}} \iint \vec{\nabla} f \, dS_d$? $\Rightarrow$ electrostatics!

single vector is defined: the dipolar vector

$\vec{O\lambda_+}$ w/ charge $\frac{1}{L}$.

$$\vec{D_2} = \frac{1}{L} \begin{cases} \frac{1}{\lambda_-} & d < R_1 - R_2 \\ -\frac{1}{\lambda_+} & d > R_1 + R_2 . \end{cases} \qquad \vec{D_1} = \frac{1}{L(\lambda_- - d)}$$

$\frac{1}{S_d}$

if you don't solve electrostatics: use Fourier decomp.

use $$\frac{1}{2\pi r}\,\vec{e_z}\cdot\oint_{C(0,r)} \nabla \log|z-a|\,dl = \begin{cases} -\frac{1}{a}, & |a|>r \\ 0, & |a|<r. \end{cases}$$

& Use capacitance to obtain growth; dipole = migration.
(d=0).

$$\begin{cases} \dfrac{dR_1}{dt} = \dfrac{M_v \varepsilon}{R\Delta\Phi}\left(1 - \boxed{\dfrac{R_{c,eff}(s)}{R}}^{\,(!)}\right) \\ \\ \underbrace{\dfrac{dR_2}{dt} = -\dfrac{2M_L\gamma_v}{R^2\Delta\Phi^2}\,\dfrac{1+s}{s^2(1-s)}}_{\text{"}\varepsilon=0\text{ result" modulated}} \end{cases} \qquad s = R_2/R_1$$

bubbles w/ $\gamma_v<0$, $\gamma_L>0$ help mediate the structure

$$R_{c,eff} = R_c\left(1 + \overbrace{\frac{M_L/\gamma_L}{M_v/\gamma_v}\;\frac{1+s}{1-s}}\right)$$

$$R_c = \frac{2\gamma}{\varepsilon\Delta\Phi}.$$ (single droplet.)