Quadratic optimizations in wave physics

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Les Gustins Summer School 2026

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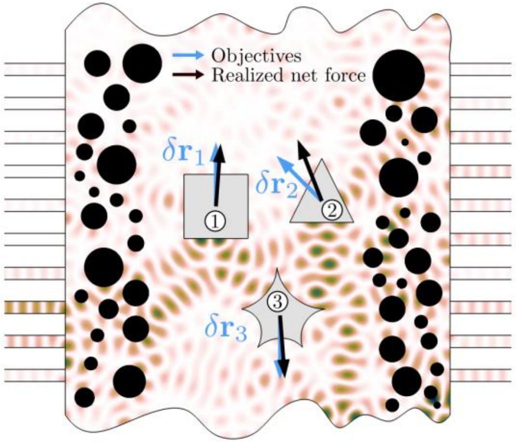
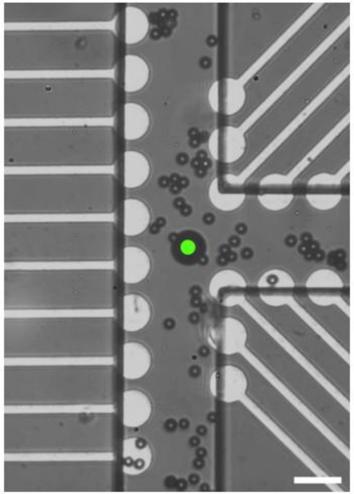
Last year: « Introduction à la matrice de Wigner-Smith généralisée pour la modulation de front d'onde »

Let’s dig deeper into the topic!





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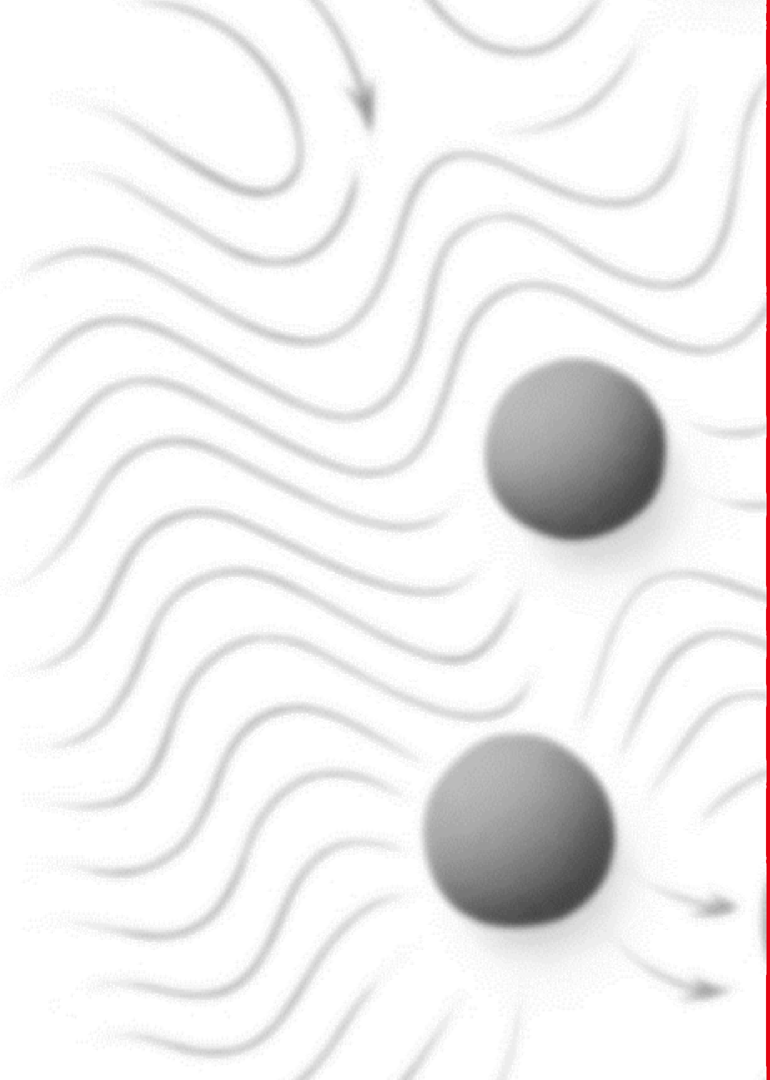
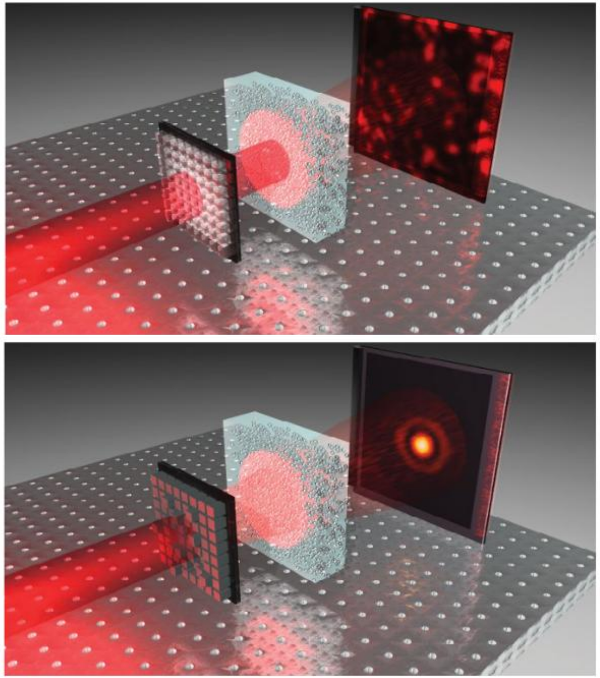


Introduction

Wavefront shaping techniques

Optimization of a wave's incident degree of freedom (typically phase and/or amplitude) to maximize a chosen objective after propagation (and scattering!)



Cao, Mosk, Rotter. Nature Physics 18.9 (2022): 994-1007

Speckle

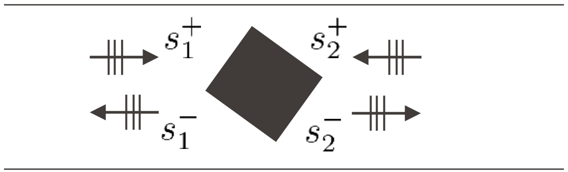
Focused beam

Iterative approaches

One-shot approaches

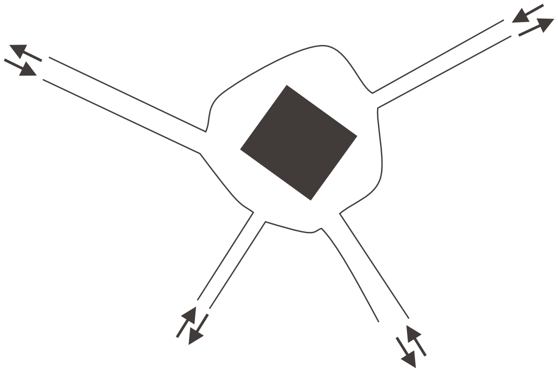
Scattering matrix: linearity assumption

Plane waves example



$$\begin{pmatrix} s_1^- \\ s_2^- \end{pmatrix} = \mathbf{S} \begin{pmatrix} s_1^+ \\ s_2^+ \end{pmatrix}$$

In general

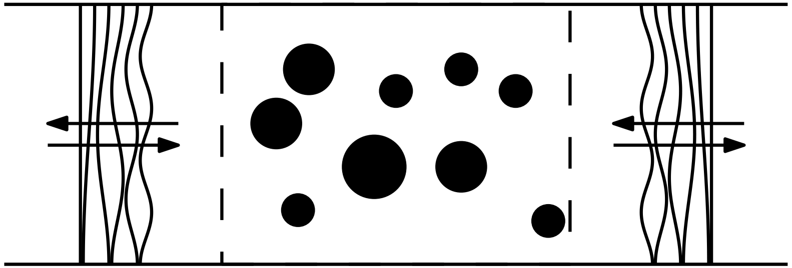


$$\mathbf{s}_- = \mathbf{S}\mathbf{s}_+$$

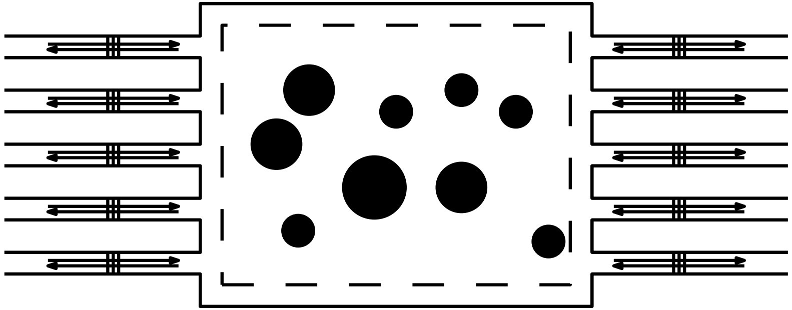
$$\underset{[(m_L+m_R)\times(m_L+m_R)]}{\mathbf{S}} = \begin{pmatrix} \underset{[m_L\times m_L]}{\mathbf{r_L}} & \underset{[m_L\times m_R]}{\mathbf{t_{R\to L}}} \\ \underset{[m_R\times m_L]}{\mathbf{t_{L\to R}}} & \underset{[m_R\times m_R]}{\mathbf{r_R}} \end{pmatrix}$$

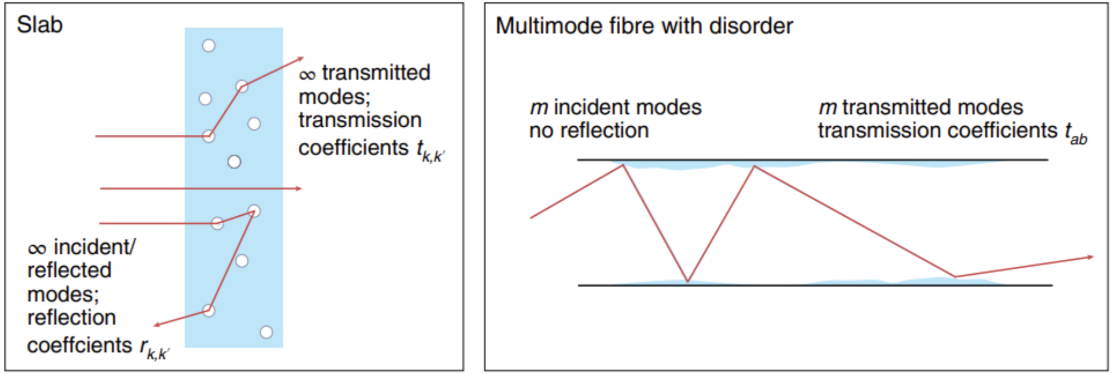
Scattering matrix: linearity assumption

Multimode waveguide



Multiport cavity with monomode waveguides





Cao, Mosk, Rotter. Nature Physics 18.9 (2022): 994-1007

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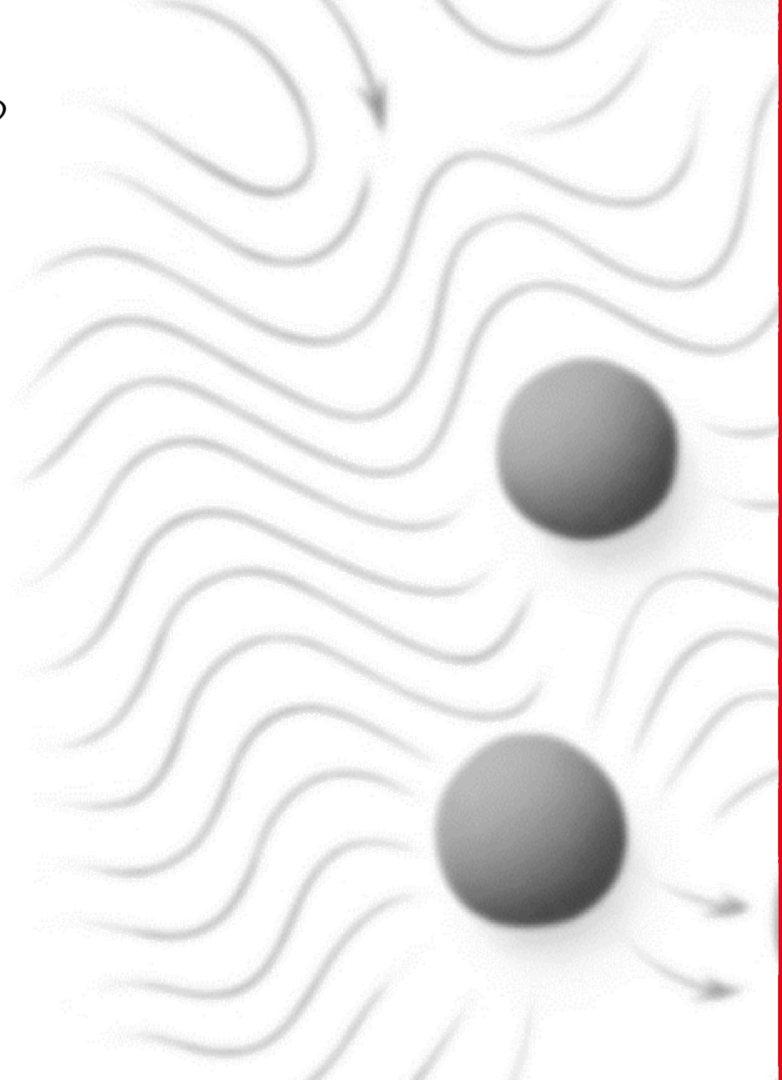
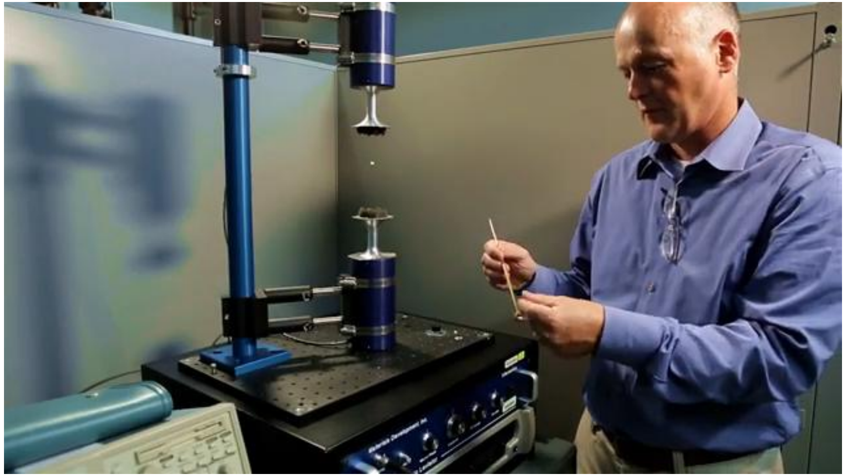


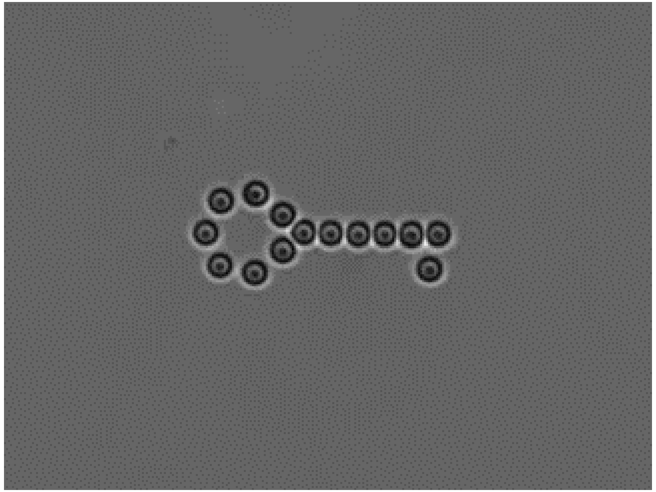
Radiation force optimization

Acoustical or Optical tweezers

→ Momentum transfer



Argonne National Laboratory



Elliot Scientific

Radiation force densities: *quadratic* quantities

Electromagnetism

- Under a certain set of approximations, from Lorentz force:

$$\bar{\mathbf{f}}_{\mathrm{em}} = -\frac{1}{4}\left(|\mathbf{E}|^2\nabla\varepsilon + |\mathbf{H}|^2\nabla\mu\right)$$

Anghinoni, Bruno, Mikko Partanen, and Nelson GC Astrath. The European Physical Journal Plus 138.11 (2023): 1034.

Acoustics

- An analogue formula accounting for momentum transfer in nonlinear acoustics

$$\bar{\mathbf{f}}_{\mathrm{ac}} = -\frac{1}{4}\left(|p_1|^2\nabla\kappa_0 + |\mathbf{v}_1|^2\nabla\rho_0\right)$$

Karlsen, Jonas T., Per Augustsson, and Henrik Bruus. Physical Review Letters 117.11 (2016): 114504

The cool stuff is that both equations have the same structure (in lossless media)

(Of course, the devil is in the details: obtaining the above equations is really non trivial)

Why maximizing momentum transfer?

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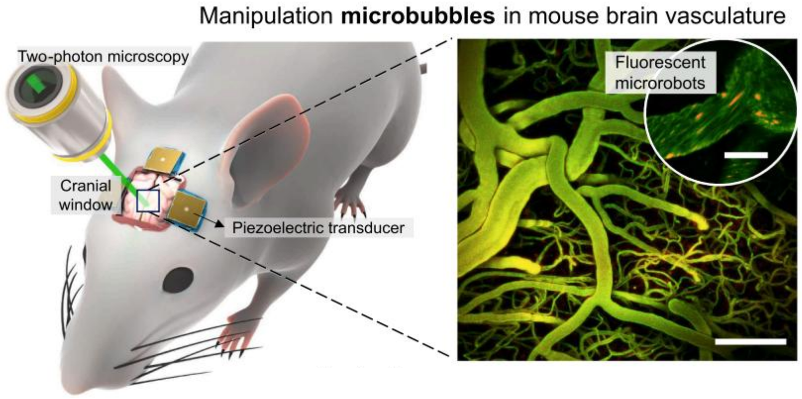
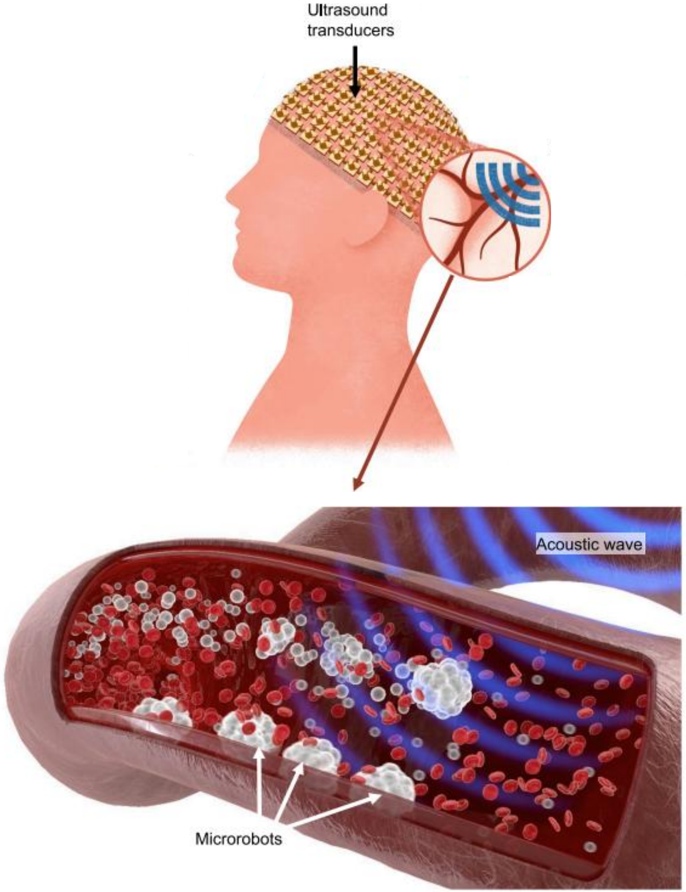
Precision therapy

Navigate microrobots through complex pathways within deep tissues or organs





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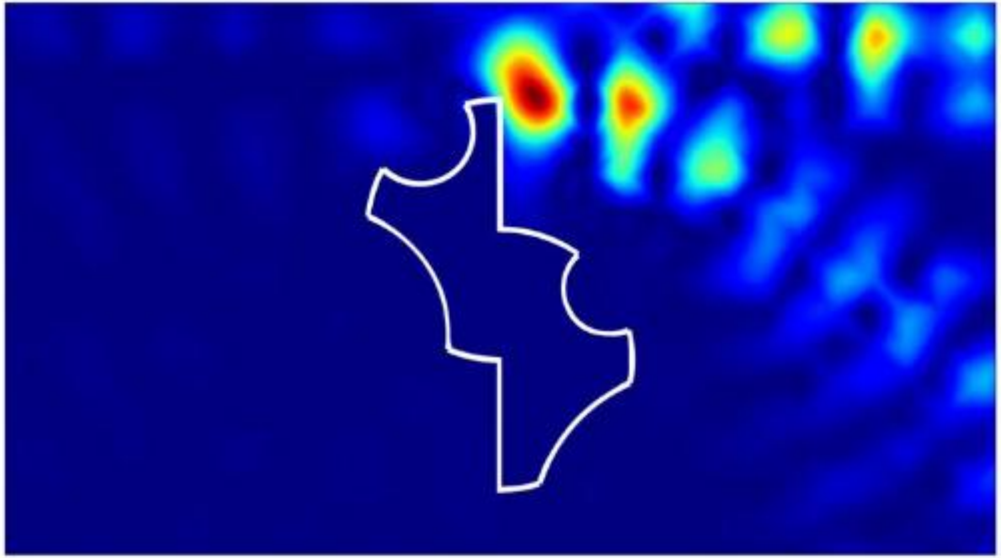
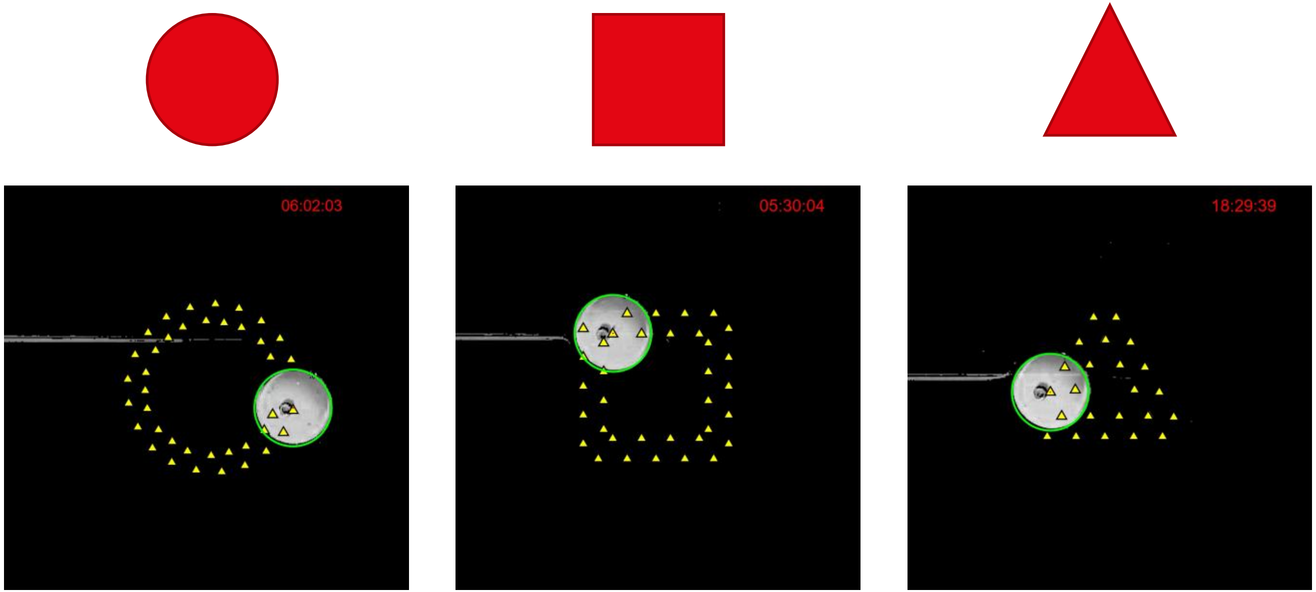
- Fonseca and Ahmed. Advanced Drug Delivery Reviews 205 (2024): 115164

Maximizing electromagnetic torque on 1 object



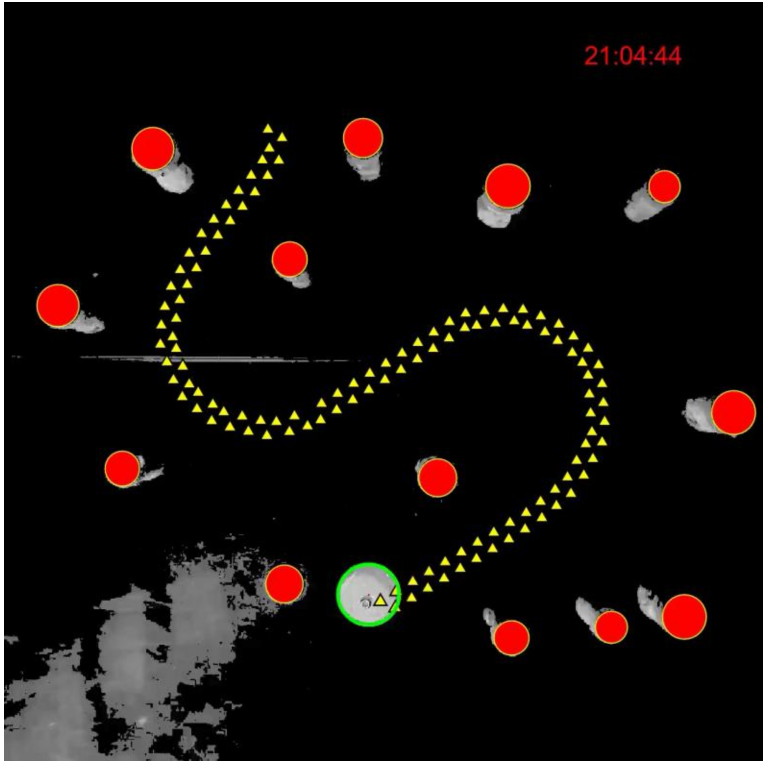
Horodynski, Michael, et al. Nature Photonics 14.3 (2020): 149-153.

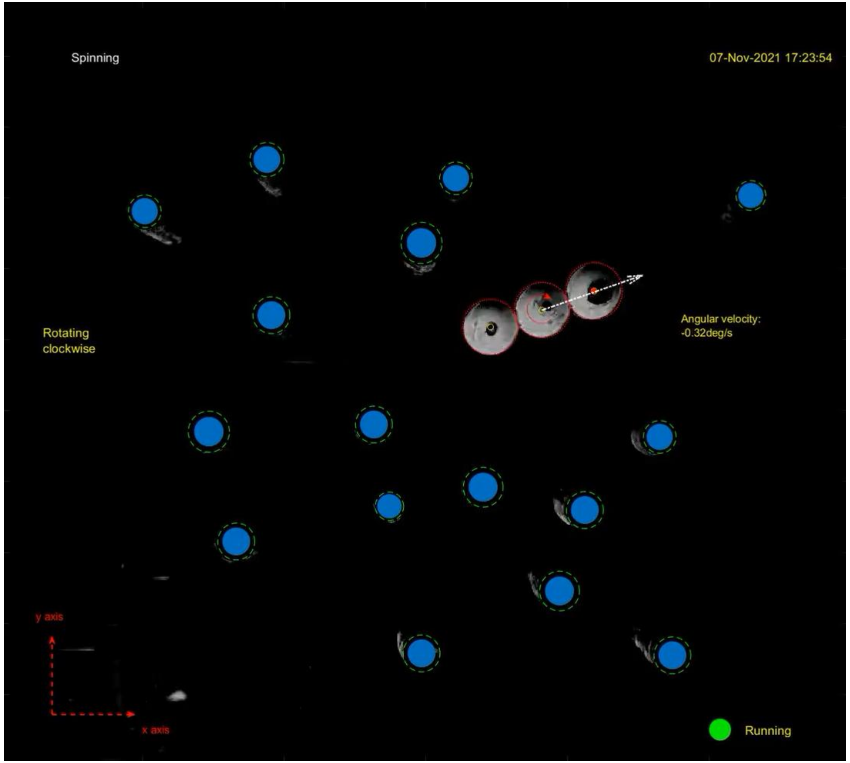
Maximizing acoustic force on 1 object



Orazbayev, et al. Nature Physics 20.9 (2024): 1441-1447

Maximizing acoustic force/torque on 1 object

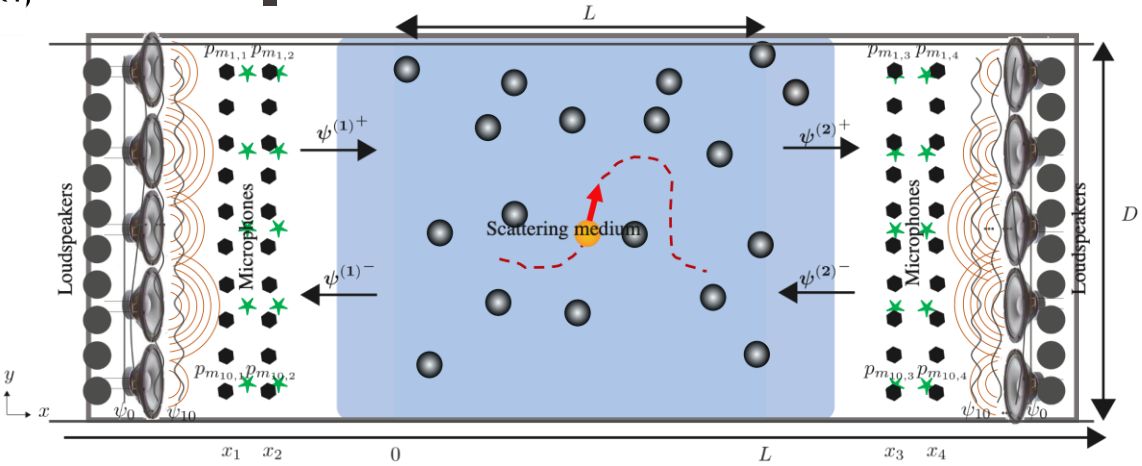


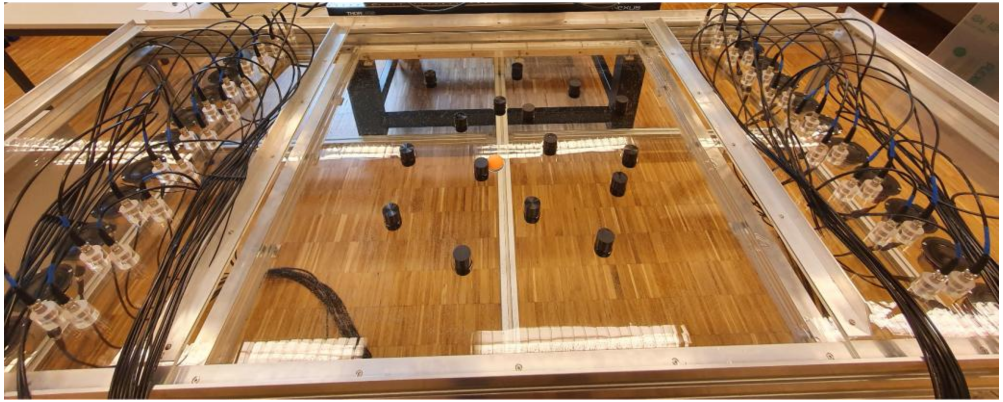


Orazbayev, et al. Nature Physics 20.9 (2024): 1441-1447

The experiment

2D multimode waveguide





Orazbayev, et al. Nature Physics 20.9 (2024): 1441-1447

Measure x_i, y_i *(camera)*

Measure **S**(x_i, y_i, t_i) *(microphones)*

Find optimal mode mixture to push the ball at position i+1 *(computer)*

Send this wavefront and let the ball move close enough to next point *(speakers)*

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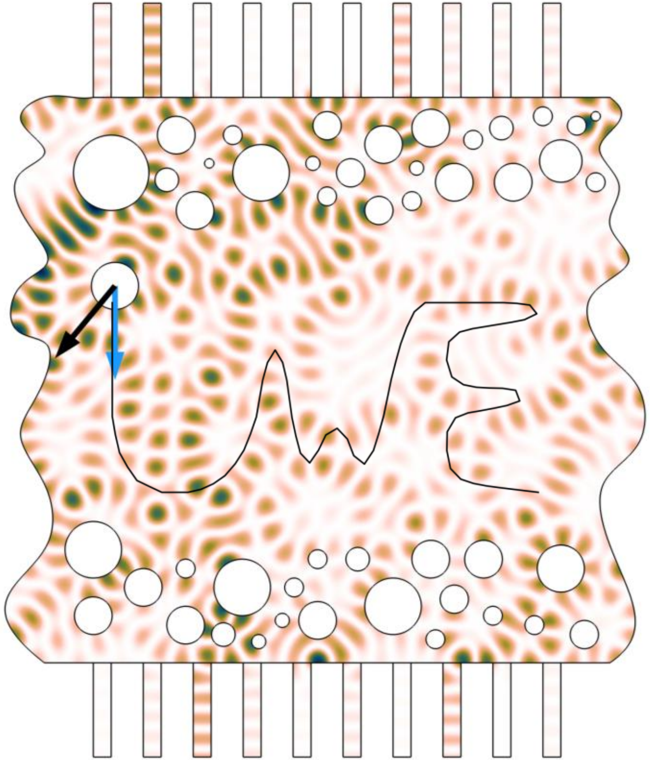
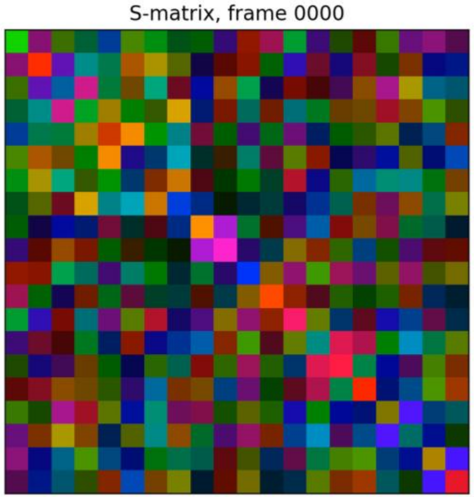
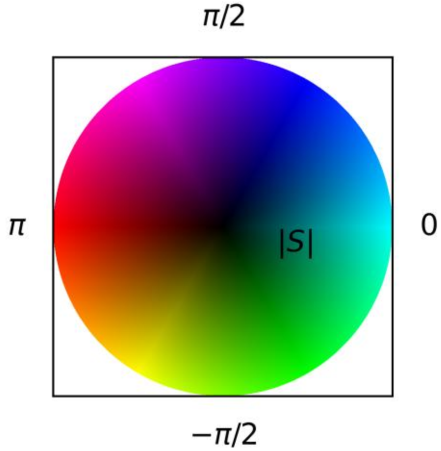
From the field's perspective







Objectives

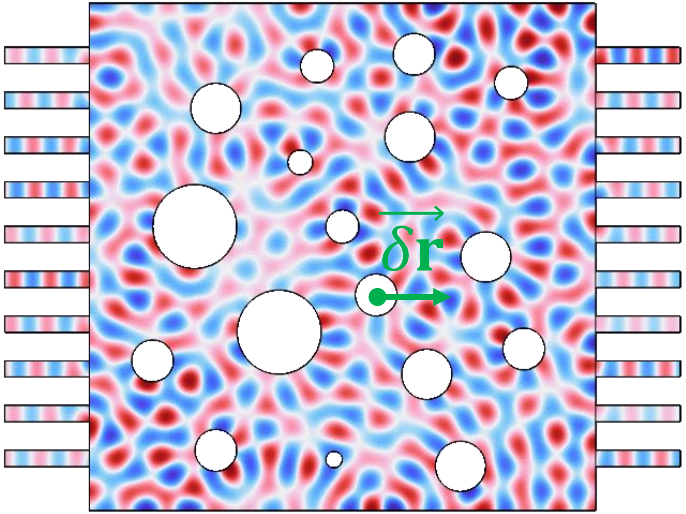
Realized net force

From radiation forces to the GWS matrix

1) Upon a virtual displacement $\overrightarrow{\delta \mathbf{r}}$, the radiation force provides the work:

$$\delta W = \vec{\mathbf{F}}.\overrightarrow{\delta \mathbf{r}}$$

2) If we have no loss, this work can be interpreted from the ports. Under fixed harmonic excitation $\mathbf{s}_+$, it manifests as a scattering change in $\mathbf{s}_- = \mathbf{S}\mathbf{s}_+$



$\mathbf{s}_-$ components: $b_n = \sqrt{P_n} e^{j\varphi_n} \rightarrow \mathbf{s}_-^\dagger \delta \mathbf{s}_- = \frac{1}{2}\sum_n \delta P_n + j \sum_n P_n \delta\varphi_n$

We can show that*:

$$\delta W = -\frac{1}{\omega}\mathrm{Im}[\mathbf{s}_-^\dagger \delta \mathbf{s}_-]$$

Using the lossless assumption, we also find that:

$$\sum_n P_n \delta\varphi_n = -j \sum_n b_n^* \delta b_n = -j\mathbf{s}_-^\dagger \delta \mathbf{s}_- = -\mathbf{s}_+^\dagger (j\mathbf{S}^\dagger \delta \mathbf{S})\mathbf{s}_+$$

Thus $\vec{\mathbf{F}}.\overrightarrow{\delta \mathbf{r}} = \delta W = \frac{1}{\omega}\mathbf{s}_+^\dagger (j\mathbf{S}^\dagger \delta \mathbf{S})\mathbf{s}_+$, or

$$\boxed{\vec{\mathbf{F}}.\overrightarrow{\delta \mathbf{r}} = \frac{1}{\omega}\mathbf{s}_+^\dagger \mathbf{Q}_{\overrightarrow{\delta \mathbf{r}}} \mathbf{s}_+ \equiv \frac{1}{\omega}\langle \mathbf{Q}_{\overrightarrow{\delta \mathbf{r}}} \rangle_{\mathbf{s}_+}}$$

$$\mathbf{Q}_{\overrightarrow{\delta \mathbf{r}}} = j\mathbf{S}^\dagger (\overrightarrow{\delta \mathbf{r}}.\nabla_{\overrightarrow{\delta \mathbf{r}}})\mathbf{S}$$

*a complete proof can be found here:
Nerson, Tristan, et al. Physical Review Research 8.2 (2026): 023254

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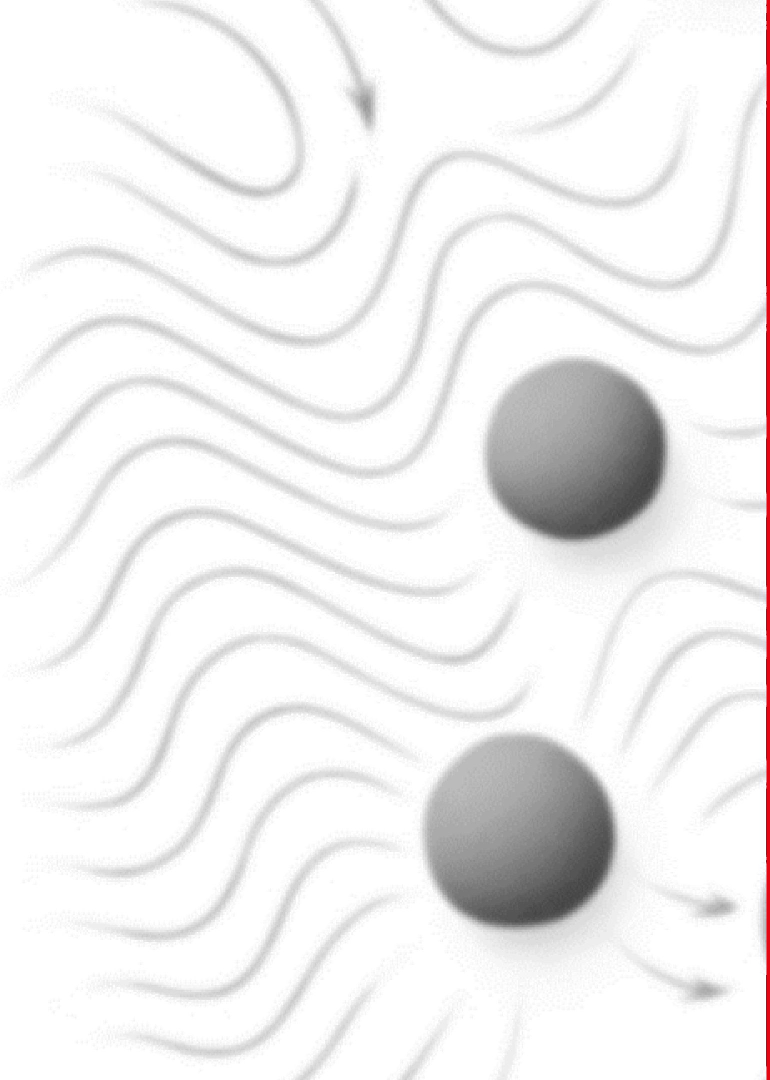


Quadratic programs

Quadratic programs

We have a quadratic program:

$$\max_{\mathbf{s}_+ \in \mathbb{C}^N} \quad \langle \mathbf{Q}_\alpha \rangle$$

$$\text{s.t.} \quad \mathbf{s}_+^\dagger \mathbf{s}_+ = 1$$

We can show it's Hermitian

Input power normalization

This is the typical Rayleigh-quotient problem, which admits the solution:

$$\max \langle \mathbf{Q}_\alpha \rangle = \lambda_N(\mathbf{Q}_\alpha)$$

$$\mathbf{s}_+^{\text{opt}} = \mathbf{v}_N(\mathbf{Q}_\alpha)$$

So we reduced a very complicated problem to a simple eigenvalue problem ☺

(Small proof with Lagrange multipliers)

$$\mathbf{Q}_\alpha = \mathbf{Q}_\alpha^\dagger$$

$$\mathcal{L}(\mathbf{s}_+, \lambda) \equiv \mathbf{s}_+^\dagger \mathbf{Q}_\alpha \mathbf{s}_+ - \lambda(\mathbf{s}_+^\dagger \mathbf{s}_+ - 1), \quad \lambda \in \mathbb{R}$$

Wirtinger differentiation: $\mathbf{s}_+$ and $\mathbf{s}_+^*$ treated as independent variables

Brandwood, David H. "A complex gradient operator and its application in adaptive array theory." IEE Proceedings F (Communications, Radar and Signal Processing). Vol. 130. No. 1. IEE, 1983.

Stationary conditions: $\nabla_{\mathbf{s}_+^*} \mathcal{L} = \mathbf{0}$ (Th3) and $\partial_\lambda \mathcal{L} = 0$

$$\nabla_{\mathbf{s}_+^*}(\mathbf{s}_+^\dagger \mathbf{Q}_\alpha \mathbf{s}_+) = \mathbf{Q}_\alpha \mathbf{s}_+ \quad \text{(17c)}$$

$$\nabla_{\mathbf{s}_+^*}(-\lambda(\mathbf{s}_+^\dagger \mathbf{s}_+ - 1)) = -\nabla_{\mathbf{s}_+^*}(\mathbf{s}_+^\dagger \lambda \mathbb{1} \mathbf{s}_+) = -\lambda \mathbb{1} \mathbf{s}_+$$

$$\nabla_{\mathbf{s}_+^*} \mathcal{L} = (\mathbf{Q}_\alpha - \lambda \mathbb{1})\mathbf{s}_+ = \mathbf{0}$$

$$1 - \mathbf{s}_+^\dagger \mathbf{s}_+ = 0$$

$$\boxed{\mathbf{s}_+^\dagger \mathbf{s}_+ = 1}$$

$$\boxed{\mathbf{Q}_\alpha \mathbf{s}_+ = \lambda \mathbf{s}_+} \xrightarrow{\ \mathbf{s}_+ = \mathbf{v}\ } \boxed{\mathbf{v}^\dagger \mathbf{Q}_\alpha \mathbf{v} = \lambda \mathbf{v}^\dagger \mathbf{v} = \lambda} \xrightarrow{\ \lambda \in \mathbb{R}\ } \boxed{\mathbf{s}_+^\star = \mathbf{v}_N(\mathbf{Q}_\alpha)}$$

$$\mathbf{v}^\dagger \mathbf{Q}_\alpha^\dagger \mathbf{v} = \lambda^\dagger = \lambda^* = \lambda$$

History and examples

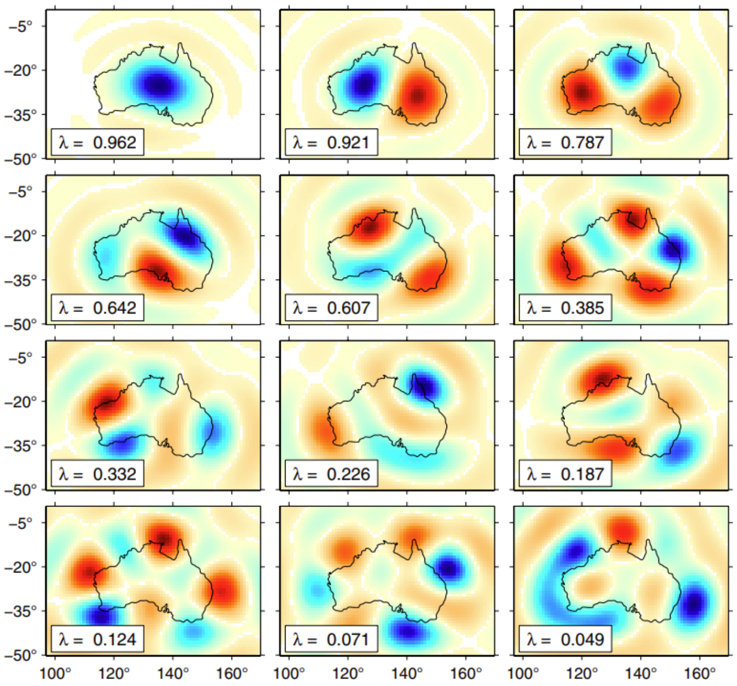
Are quadratic programs only used for radiation forces in wave physics?

Of course not!

Spatial Slepian functions

Theoretical concept: the most localized fields allowed by a spatial bandwidth limit

Min. wavelength ~ 2200 km



Simons, Dahlen, Wieczorek SIAM review 48.3 (2006): 504-536

$$\lambda = \frac{\|g\|_R^2}{\|g\|_\Omega^2} = \frac{\int_R g^2 \, d\Omega}{\int_\Omega g^2 \, d\Omega} = \frac{\mathbf{g}^T \mathbf{D} \mathbf{g}}{\mathbf{g}^T \mathbf{g}}$$

g: vector of spherical-harmonic coefficients that generates a spatial function → limited in the number of coefficents (361)

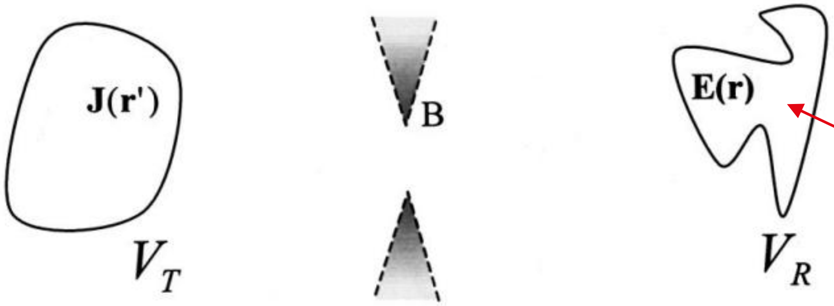
D: spatial overlap matrix between pairs of band-limited spherical harmonics inside Australia → constructive interference

Original Slepian problem: among signals strictly band-limited in frequency, maximize the fraction of energy contained within a finite time interval

Slepian and Pollak. Bell System Technical Journal 40.1 (1961): 43-63

Green function and quadratic forms

Can we maximize intensity in a zone from sources in another?



Piestun and Miller. JOSA A 17.5 (2000): 892-902

$$\mathbf{E}_i = \mathbf{G}\mathbf{J}_i$$

i → source index

$$\mathbf{J} = \sum_j c_j \mathbf{J}_j$$

$$\mathbf{G}\mathbf{J} = \sum_j c_j \mathbf{E}_j$$

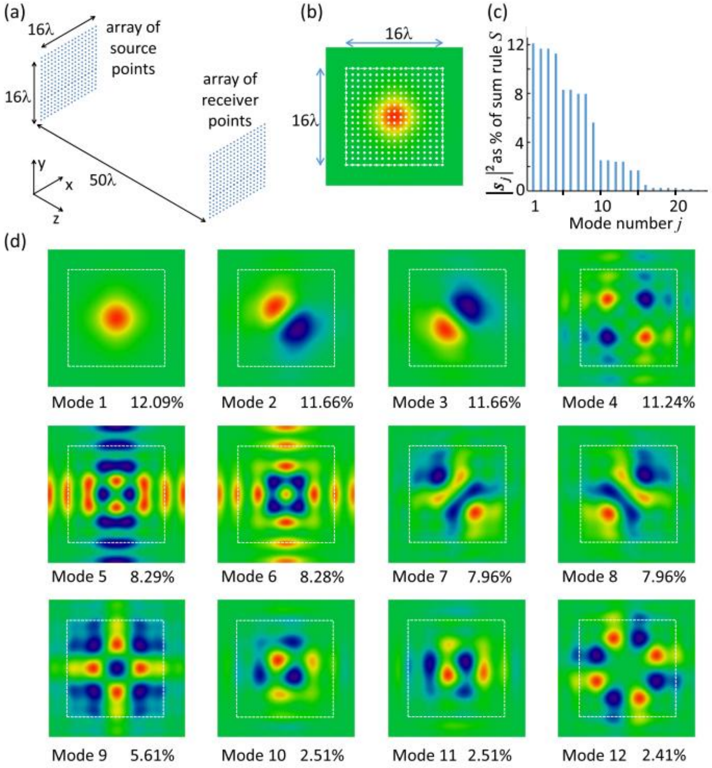
Integrated received field intensity $$||\mathbf{G}\mathbf{J}||^2_{V_\mathrm{R}} = \mathbf{c}^\dagger \mathbf{M} \mathbf{c}$$

$$||\mathbf{J}||^2_{V_\mathrm{T}} = 1$$

$$M_{ij} = \int_{V_\mathrm{R}} \mathbf{E}_i^*(\mathbf{r}) \cdot \mathbf{E}_j(\mathbf{r})\,\mathrm{d}\mathbf{r}$$

Green function and quadratic forms

Can we maximize intensity in a zone from sources in another?



Miller, David AB. Advances in Optics and Photonics 11.3 (2019): 679-825

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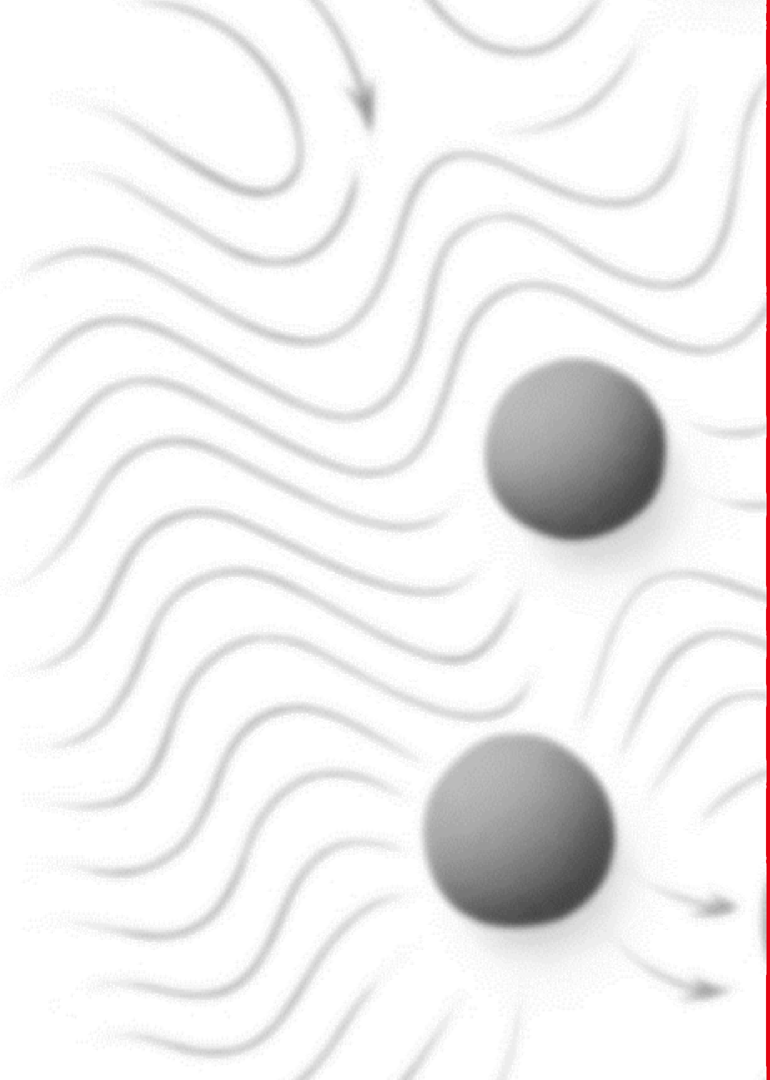

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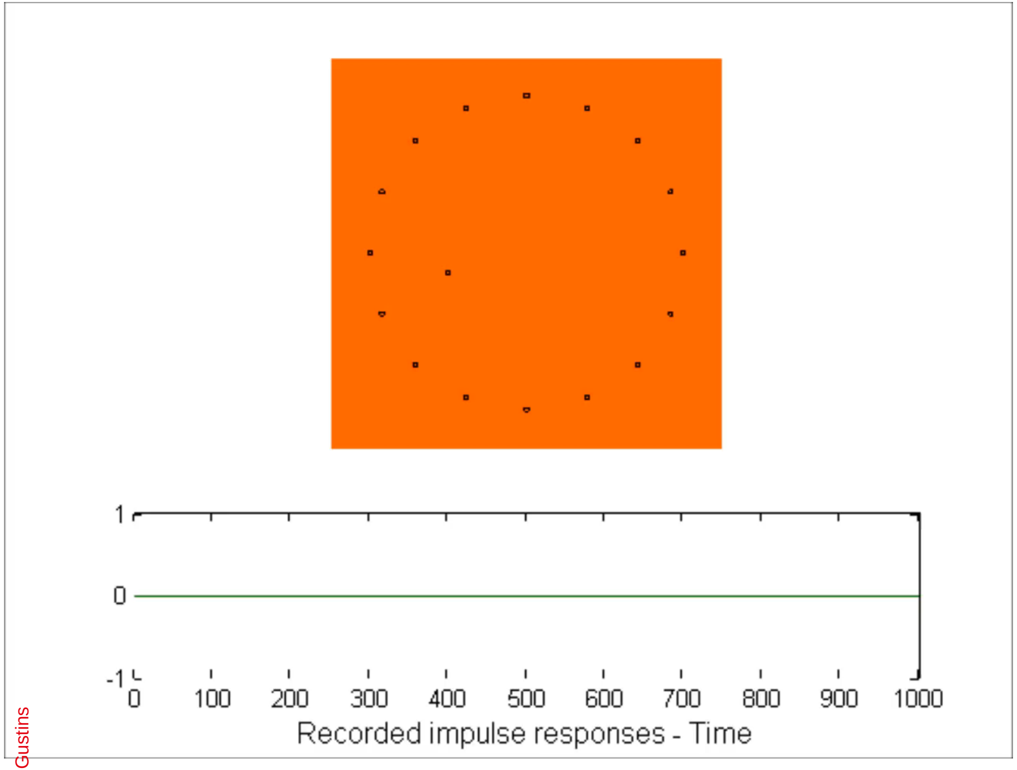
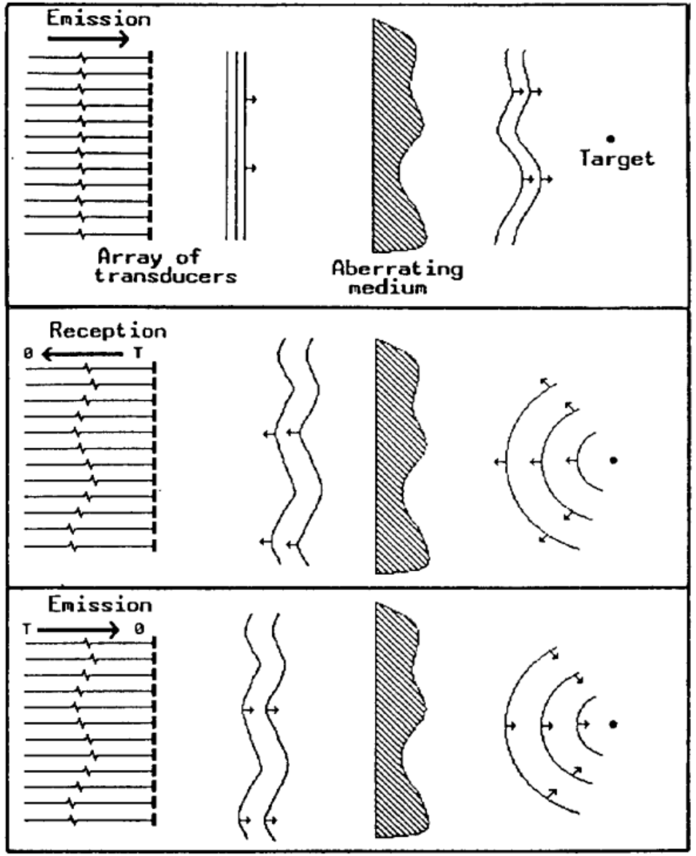
An important example of eigenvalue problem in wave physics

The French touch

Time reversal mirrors (or phase conjugation mirrors)



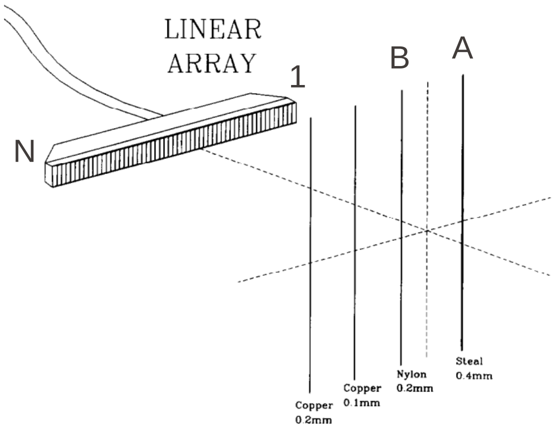
Thomas Fromentèze – FDTD simulation

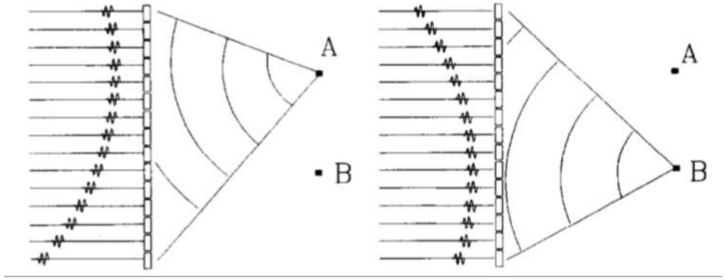


Fink, et al. Proceedings., IEEE Ultrasonics Symposium,. IEEE, 1989

The French touch

« Décomposition de l'opérateur de retournement temporel » (DORT)

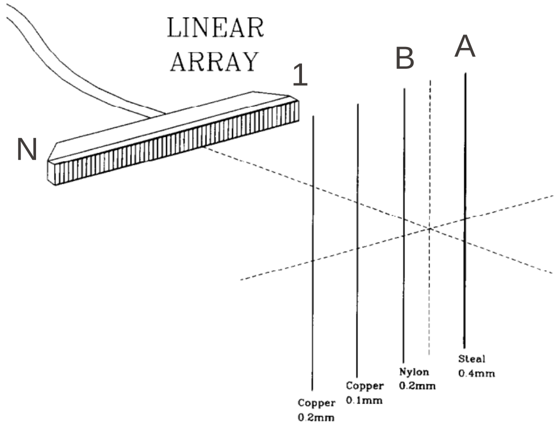




Prada and Fink. Wave motion 20.2 (1994): 151-163

The French touch

« Décomposition de l'opérateur de retournement temporel » (DORT)



$$\mathbf{h}_A = \begin{pmatrix} h_{A1} \\ \vdots \\ h_{AN} \end{pmatrix}$$

$$h_{Am} = G(\mathbf{r}_A, \mathbf{x}_m)$$

With Born approximation:

$$p_A = \mathbf{h}_A^T \mathbf{e}$$

Prop. in B after TR from A: $\mathbf{h}_A^\dagger \mathbf{h}_B = \sum_m G(\mathbf{r}_B, \mathbf{x}_m) G^*(\mathbf{r}_A, \mathbf{x}_m)$

≈ 0 For well-separated ponctual targets

$$q_A = c_A p_A = c_A \mathbf{h}_A^T \mathbf{e}$$

$$\mathbf{r}_A = c_A \mathbf{h}_A \mathbf{h}_A^T \mathbf{e} \qquad \mathbf{r}_B = c_B \mathbf{h}_B \mathbf{h}_B^T \mathbf{e}$$

$$\mathbf{r} = \mathbf{r}_A + \mathbf{r}_B = \underbrace{(c_A \mathbf{h}_A \mathbf{h}_A^T + c_B \mathbf{h}_B \mathbf{h}_B^T)}_{\mathbf{K}} \mathbf{e}$$

Pulse-echo matrix: $\mathbf{r}_0 = \mathbf{K}\mathbf{e}_0$

Time reversed: $\mathbf{e}_1 = \mathbf{r}_0^* = \mathbf{K}^* \mathbf{e}_0^* \implies \mathbf{r}_1 = \mathbf{K}\mathbf{e}_1$

Again: $\mathbf{e}_2 = \mathbf{r}_1^* = \mathbf{K}^* \mathbf{e}_1^* = \mathbf{K}^* \mathbf{K} \mathbf{e}_0$

If reciprocal: $\mathbf{K} = \mathbf{K}^T \implies \mathbf{K}^* = \mathbf{K}^\dagger$

So: $\mathbf{e}_2 = \mathbf{K}^\dagger \mathbf{K} \mathbf{e}_0$

$$\mathbf{K}\mathbf{h}_A^* = c_A \mathbf{h}_A \mathbf{h}_A^T \mathbf{h}_A^* + c_B \mathbf{h}_B \cancel{\mathbf{h}_B^T \mathbf{h}_A^*} \approx c_A ||\mathbf{h}_A||^2 \mathbf{h}_A$$

$$\mathbf{K}^\dagger \mathbf{K} \mathbf{h}_A^* \approx |c_A|^2 ||\mathbf{h}_A||^4 \mathbf{h}_A^*$$

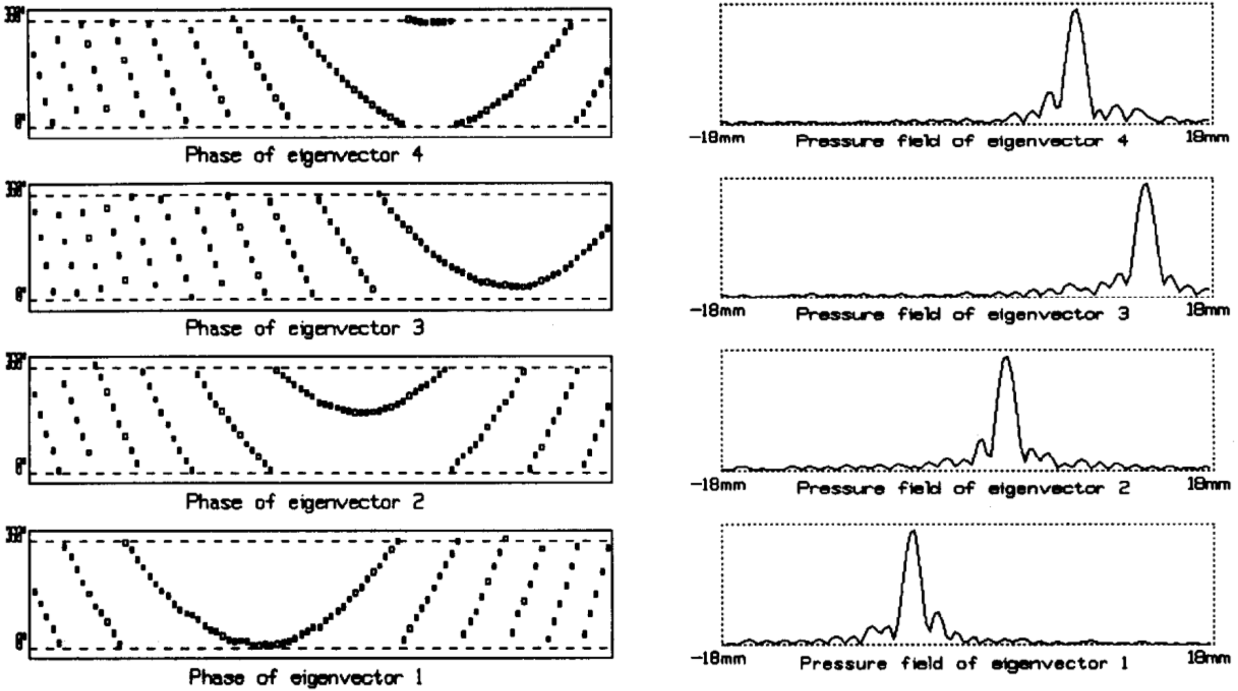
$$\mathbf{K}^\dagger \mathbf{K} \mathbf{h}_B^* \approx |c_B|^2 ||\mathbf{h}_B||^4 \mathbf{h}_B^*$$

Eigenvectors

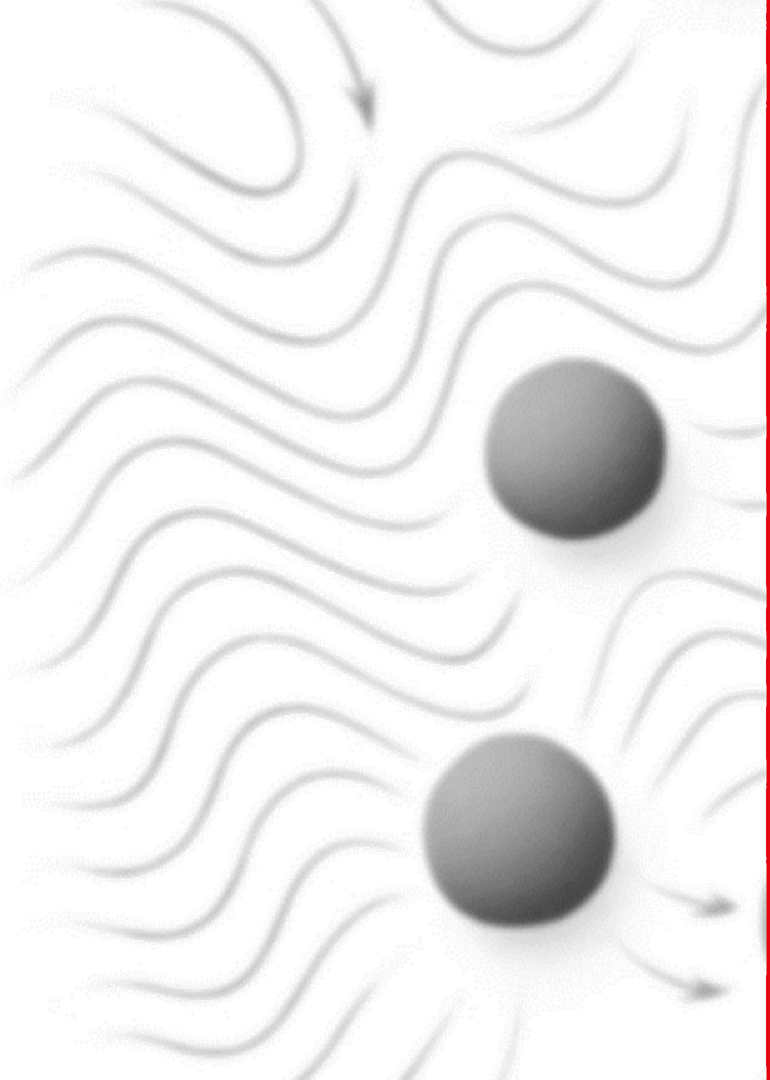
Prada and Fink. Wave motion 20.2 (1994): 151-163

The French touch

« Décomposition de l’opérateur de retournement temporel » (DORT)



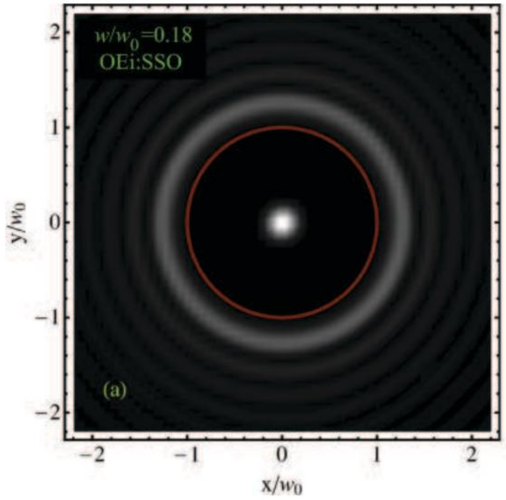
Prada and Fink. Wave motion 20.2 (1994): 151-163



BREAK

“Optical Eigenmodes”

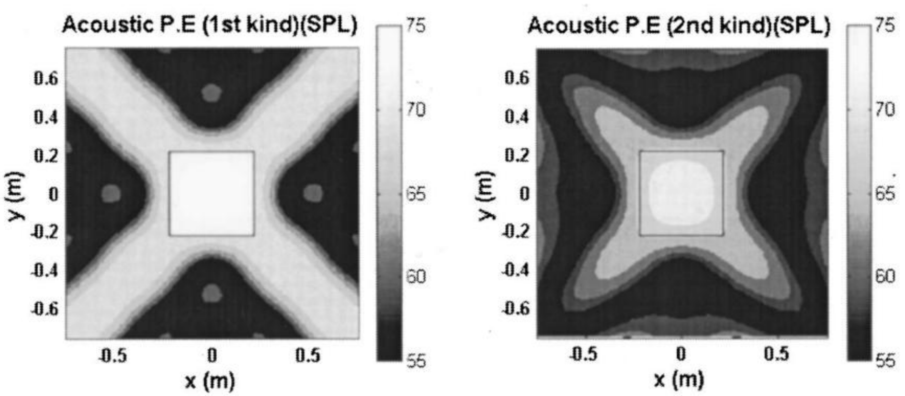
Theoretical concept: coherent optical field whose controllable input coefficients form an eigenvector of an operator representing a chosen optical observable or interaction.



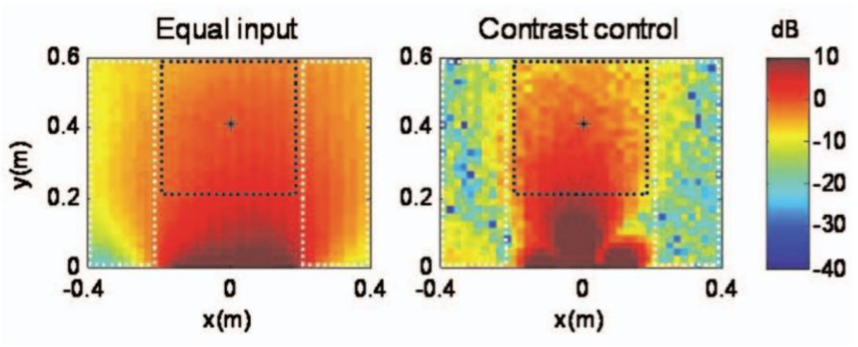
Mazilu, Baumgartl, Kosmeier, Dholakia (2011). Optics express, 19(2), 933-945

Energy, intensity, spot size, momentum,… But also more complex operations!

An older acoustic contrast example



Choi and Yang-Hann Kim. JASA 111.4 (2002): 1695-1700



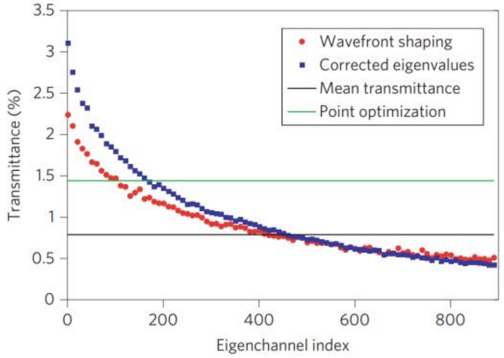
Chang, et al. JASA 125.4 (2009): 2091-2097

Examples of experiments: modern wavefront shaping

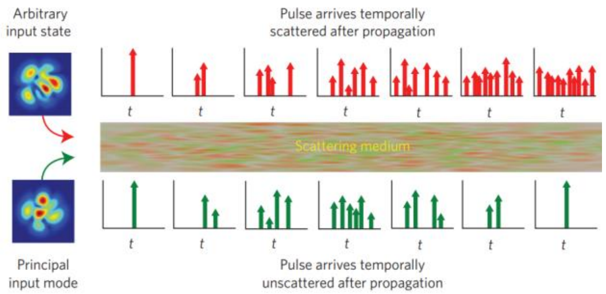
The major difference is that now, we use the scattering matrix formalism to fully account for multiple scattering in highly inhomogeneous/turbid media

→ **The theory of quadratic problems has not changed**
→ **The experiments are way more advanced**

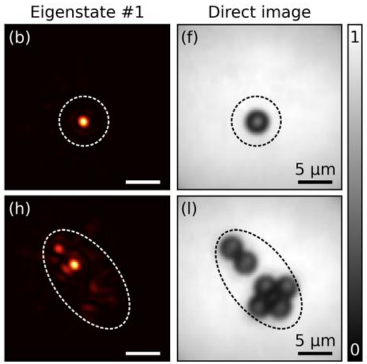
Examples of experiments: modern wavefront shaping



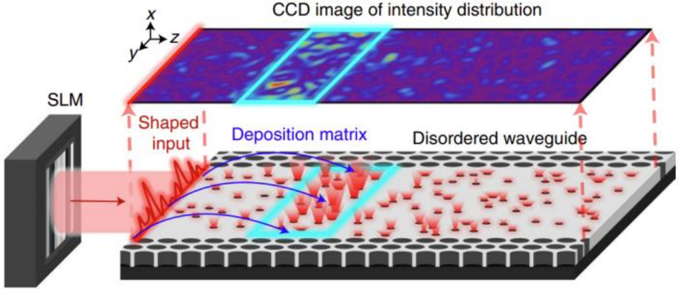
Kim, et al. Nature photonics 6.9 (2012): 581-585.



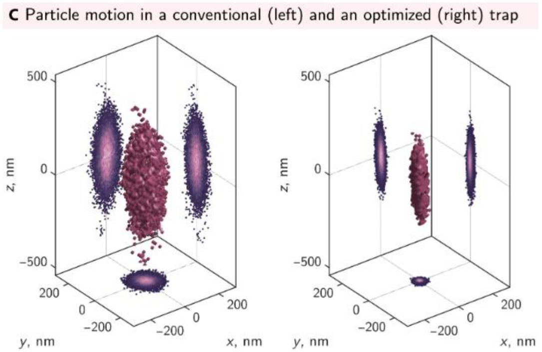
Carpenter, Eggleton, Schröder. Nature Photonics 9.11 (2015): 751-757



Bouchet, et al. Physical Review Letters 127.25 (2021): 253902

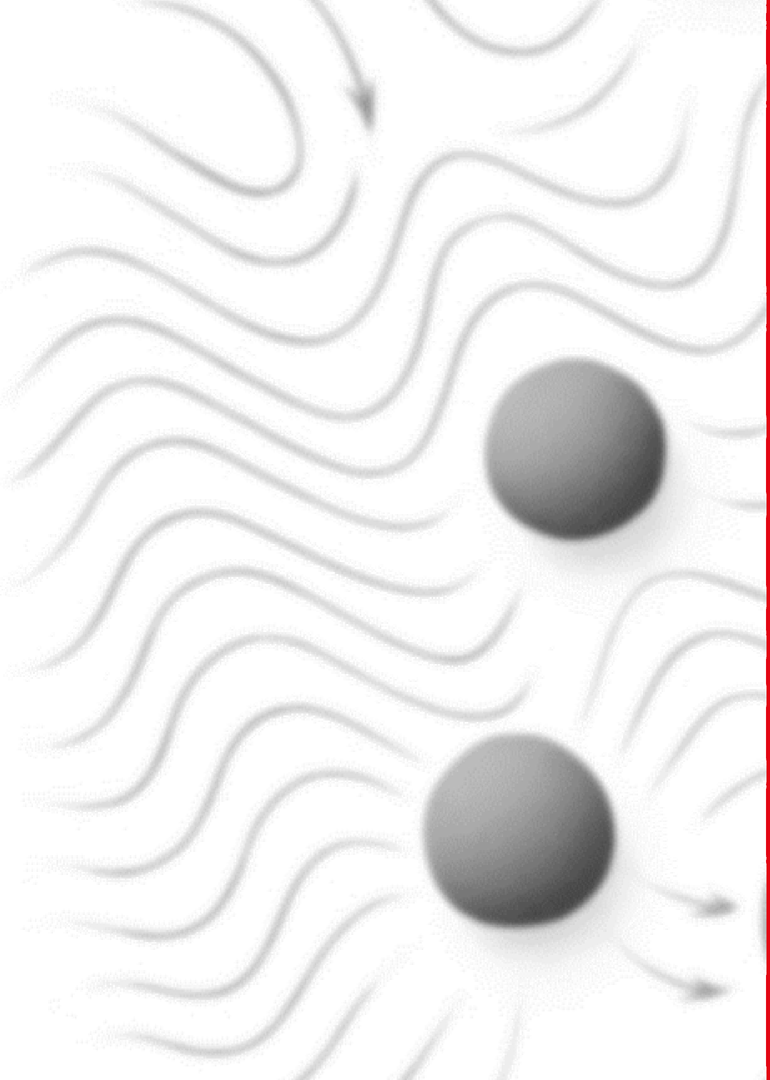


Bender, et al. Nature Physics 18.3 (2022): 309-315



Būtaitė, et al Science Advances 10.27 (2024): eadi7792

Tristan Nerson



Multi-objective optimization

Pareto optimization

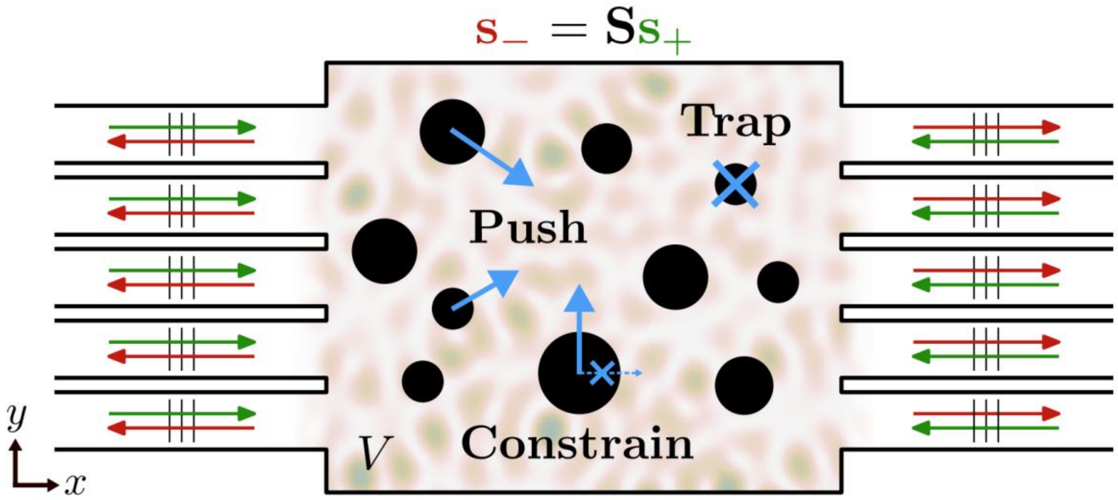
- Multi-objective optimization: $$\min_{x \in X} (f_1(x), f_2(x), \ldots, f_k(x))$$
- "A solution is called […] **Pareto optimal** if none of the objective functions can be improved in value without degrading some of the other objective values." (Wikipedia)

Pareto optimization

Tristan Nerson

Les Gustins

More complex optimization: how hard can it be?



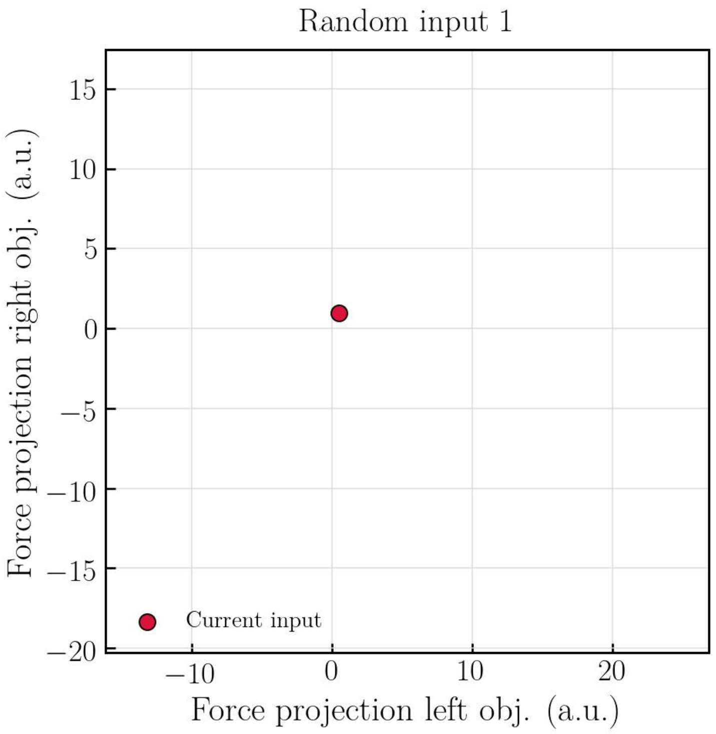
1) An objective and constraints — *"I want to maximize the radiation forces on two objects in two desired directions."*

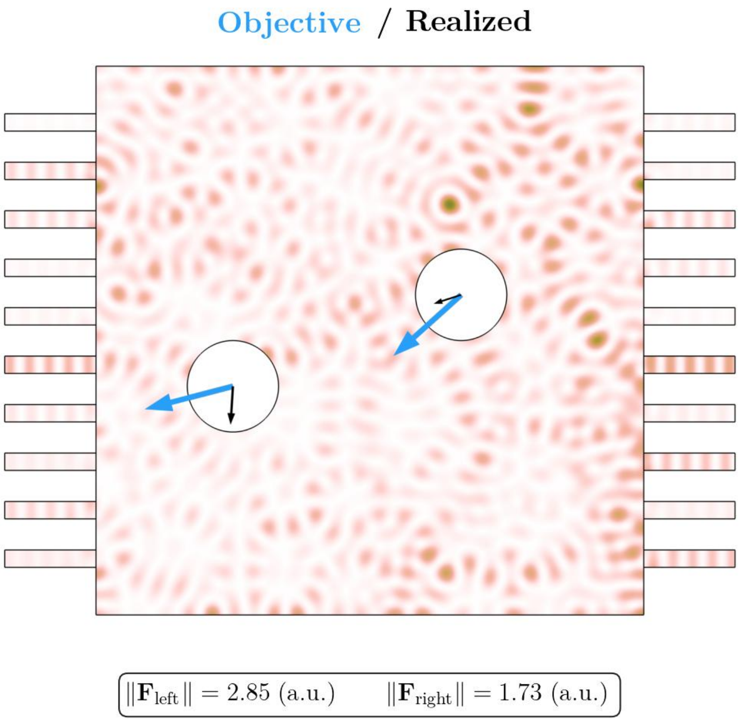
2) Wave degrees of freedom — *"I can control the amplitude and phase of N incident waves."*

3) An optimization scheme — *"I don't know... Let's try first with **random wavefronts**?"*

Nerson, Tristan, et al. Physical Review Research 8.2 (2026): 023254

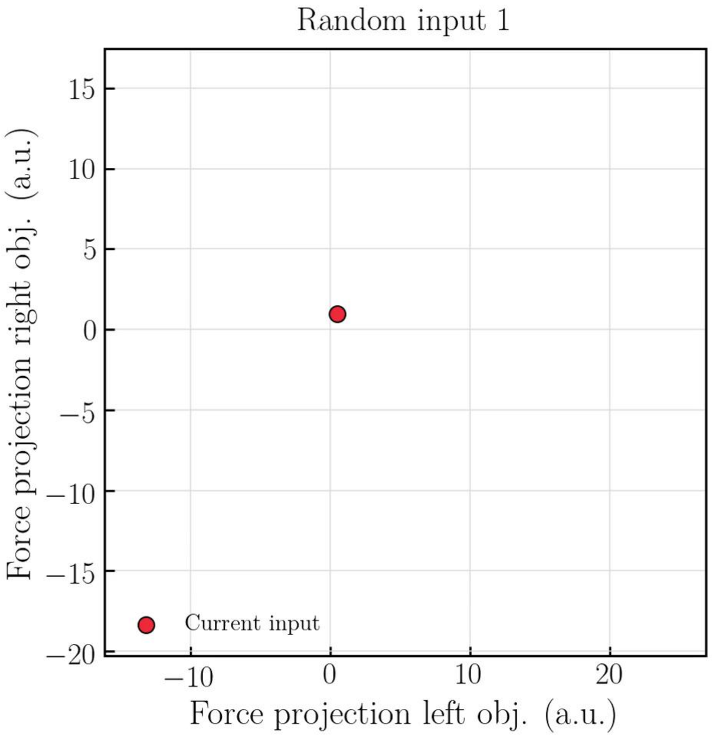
Let’s try with random wavefronts

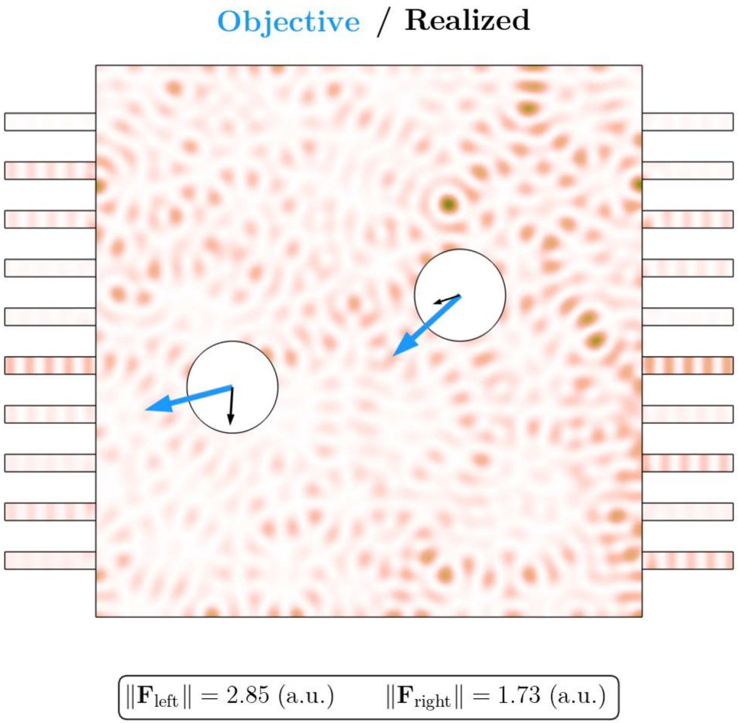




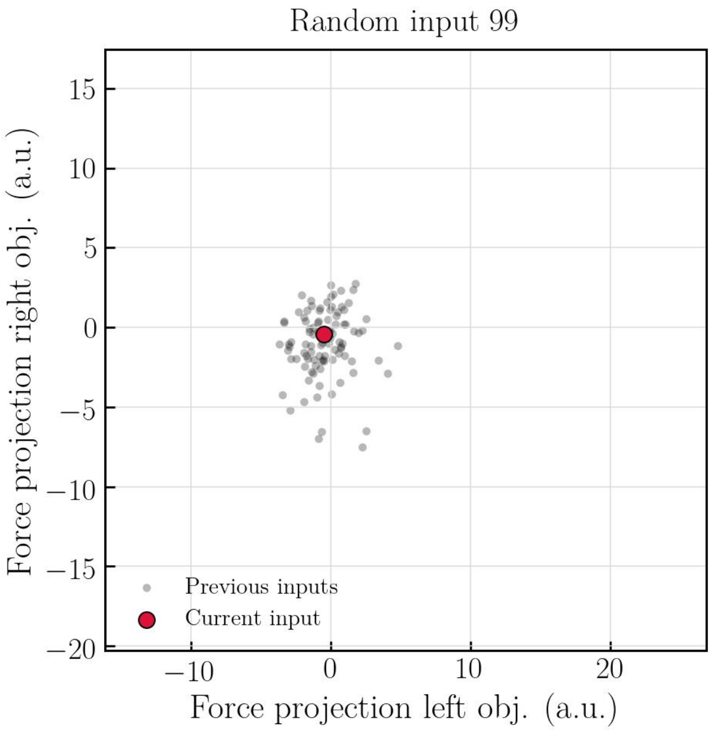
Let's try with random wavefronts

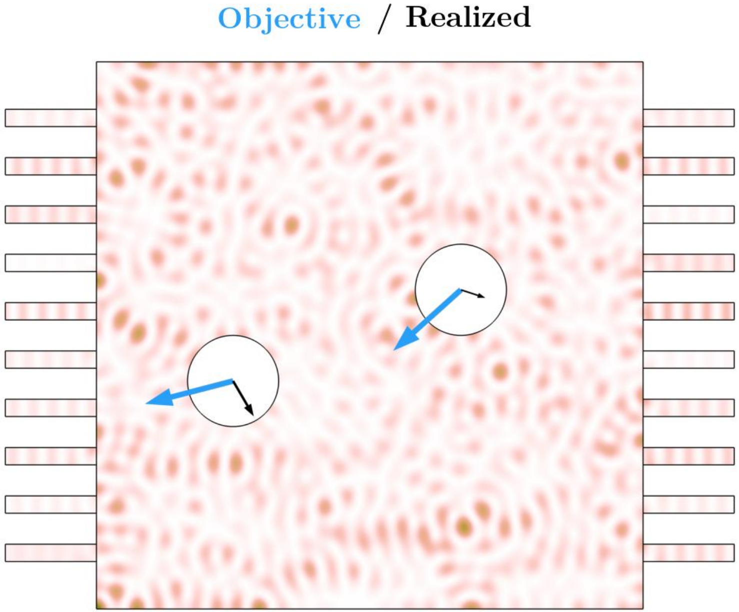
Tristan Nerson





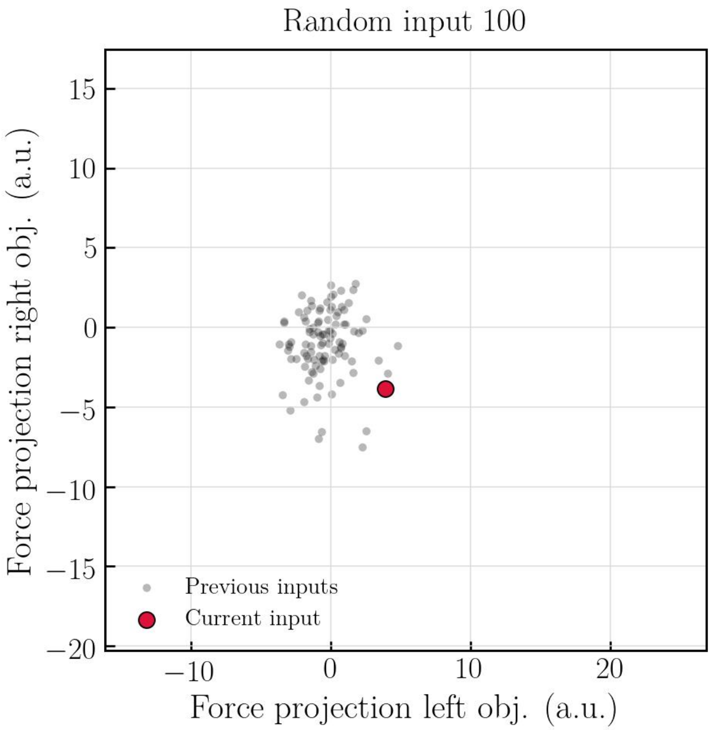
Let's try with random wavefronts

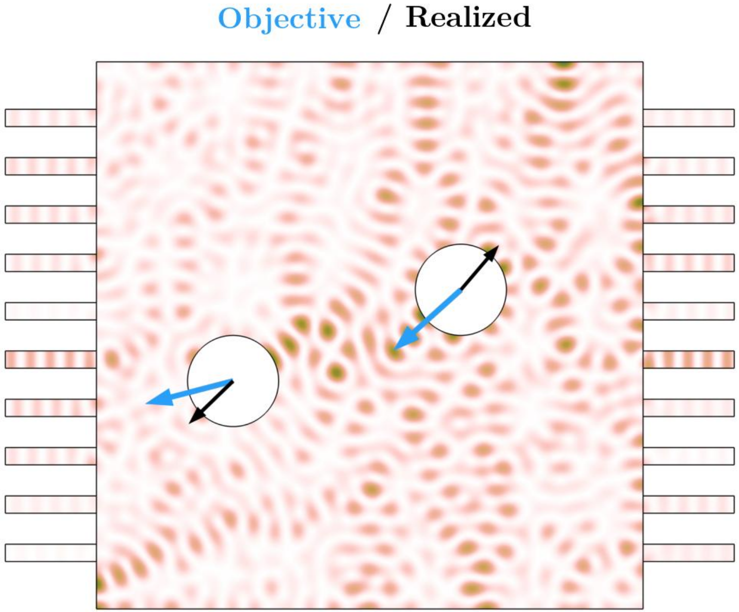




$\|\mathbf{F}_{\text{left}}\| = 3.24$ (a.u.) $\|\mathbf{F}_{\text{right}}\| = 1.44$ (a.u.)

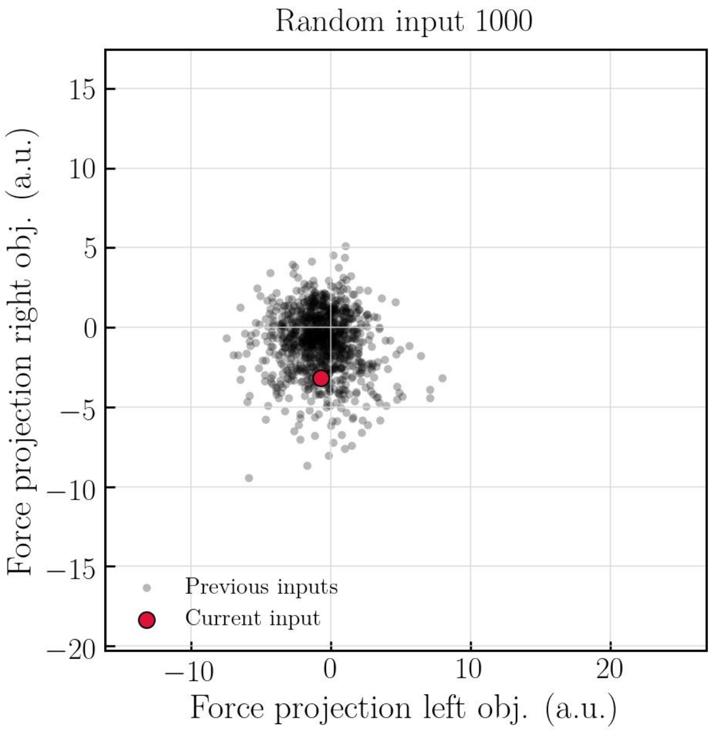
Let's try with random wavefronts

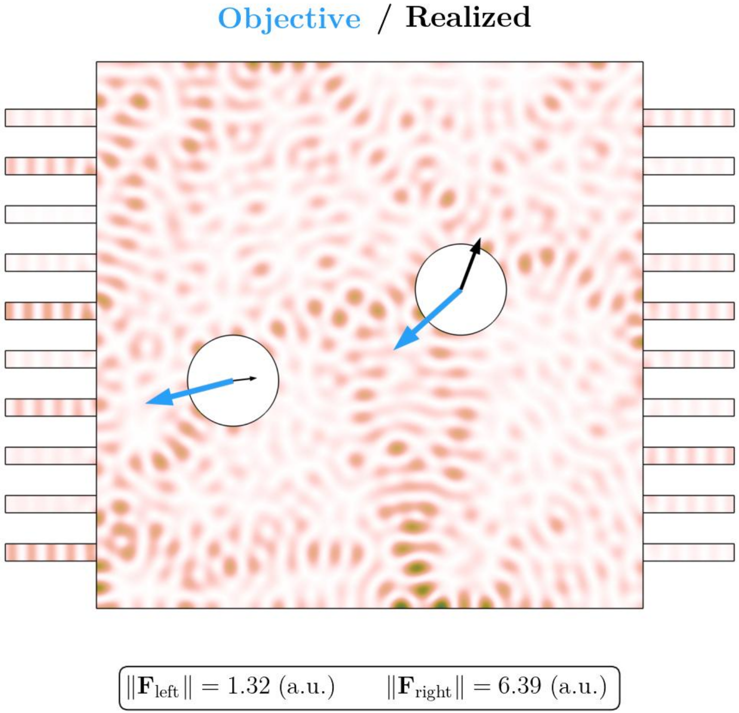




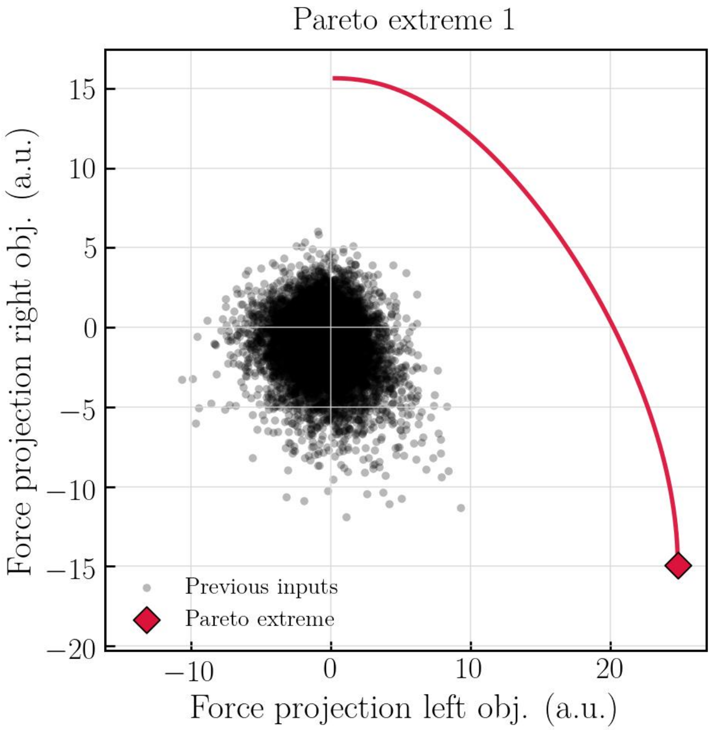
$\|\mathbf{F}_{\text{left}}\| = 8.21$ (a.u.) $\|\mathbf{F}_{\text{right}}\| = 7.19$ (a.u.)

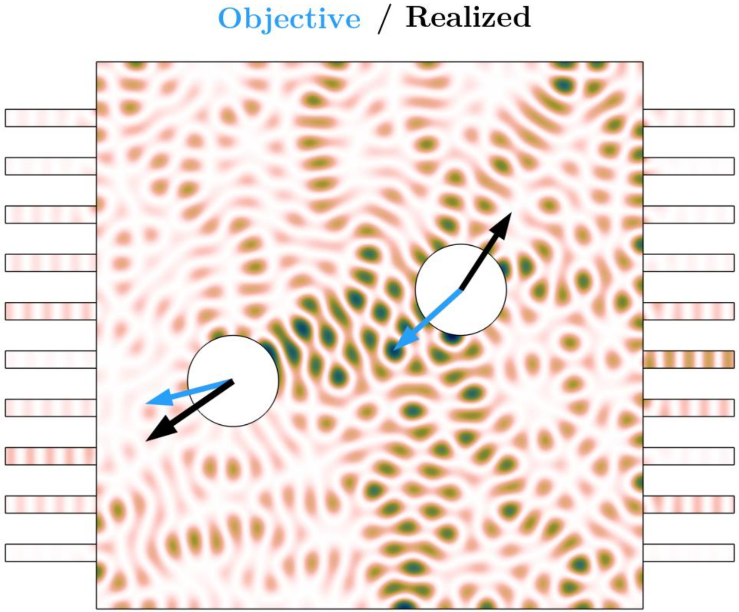
Let's try with random wavefronts





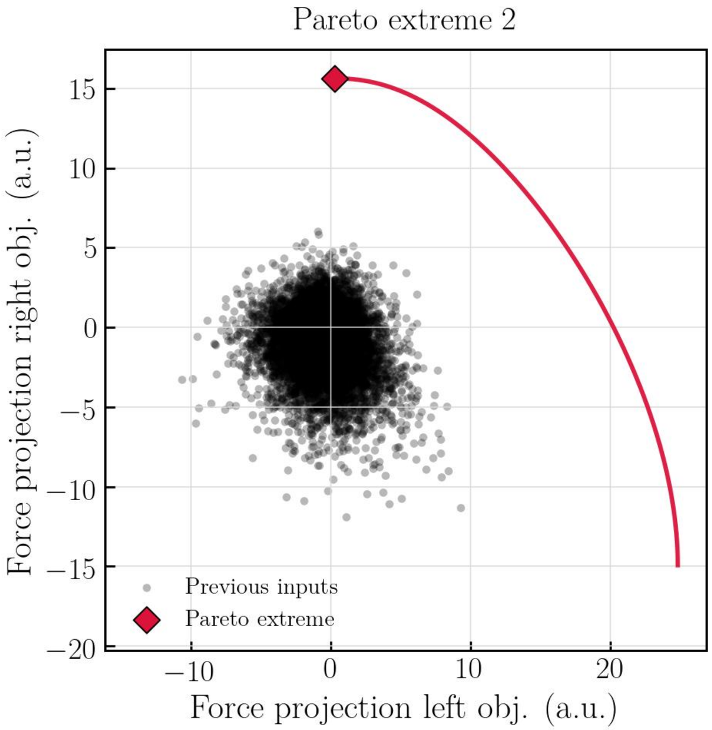
And now optimized wavefronts!

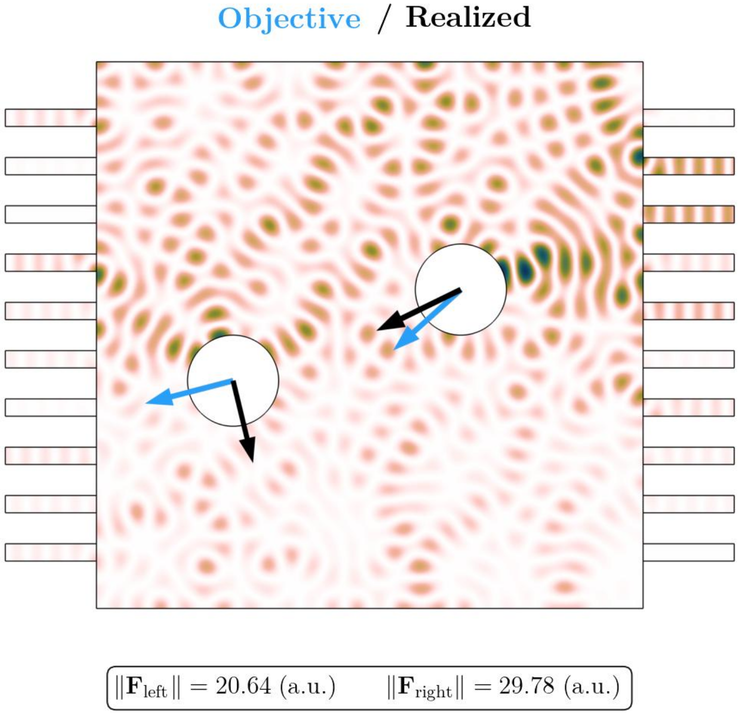




$\|\mathbf{F}_{\text{left}}\| = 48.14$ (a.u.) $\|\mathbf{F}_{\text{right}}\| = 28.45$ (a.u.)

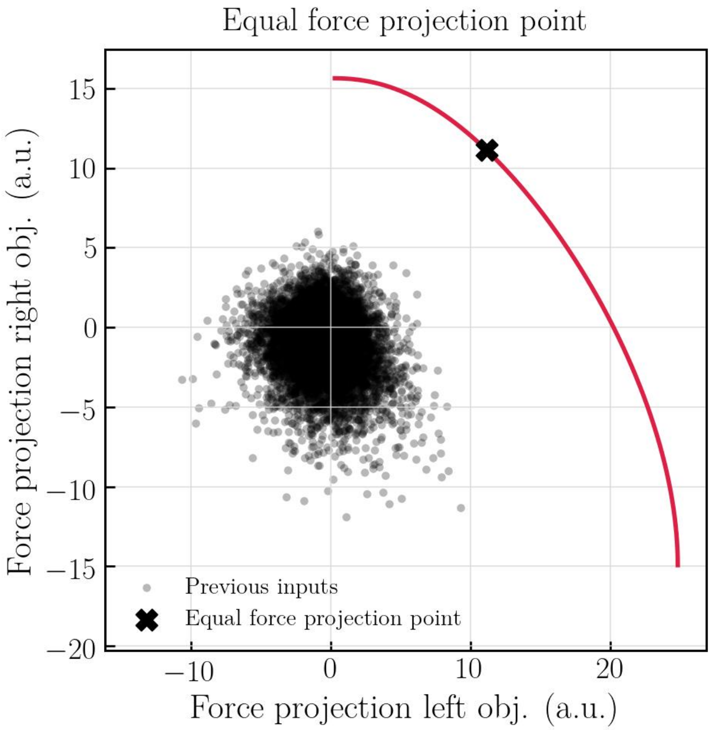
And now optimized wavefronts!

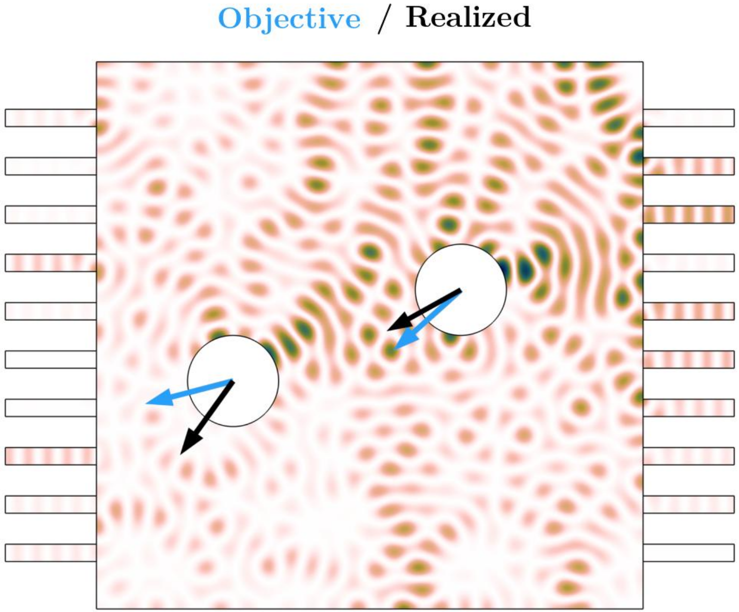




And now optimized wavefronts!

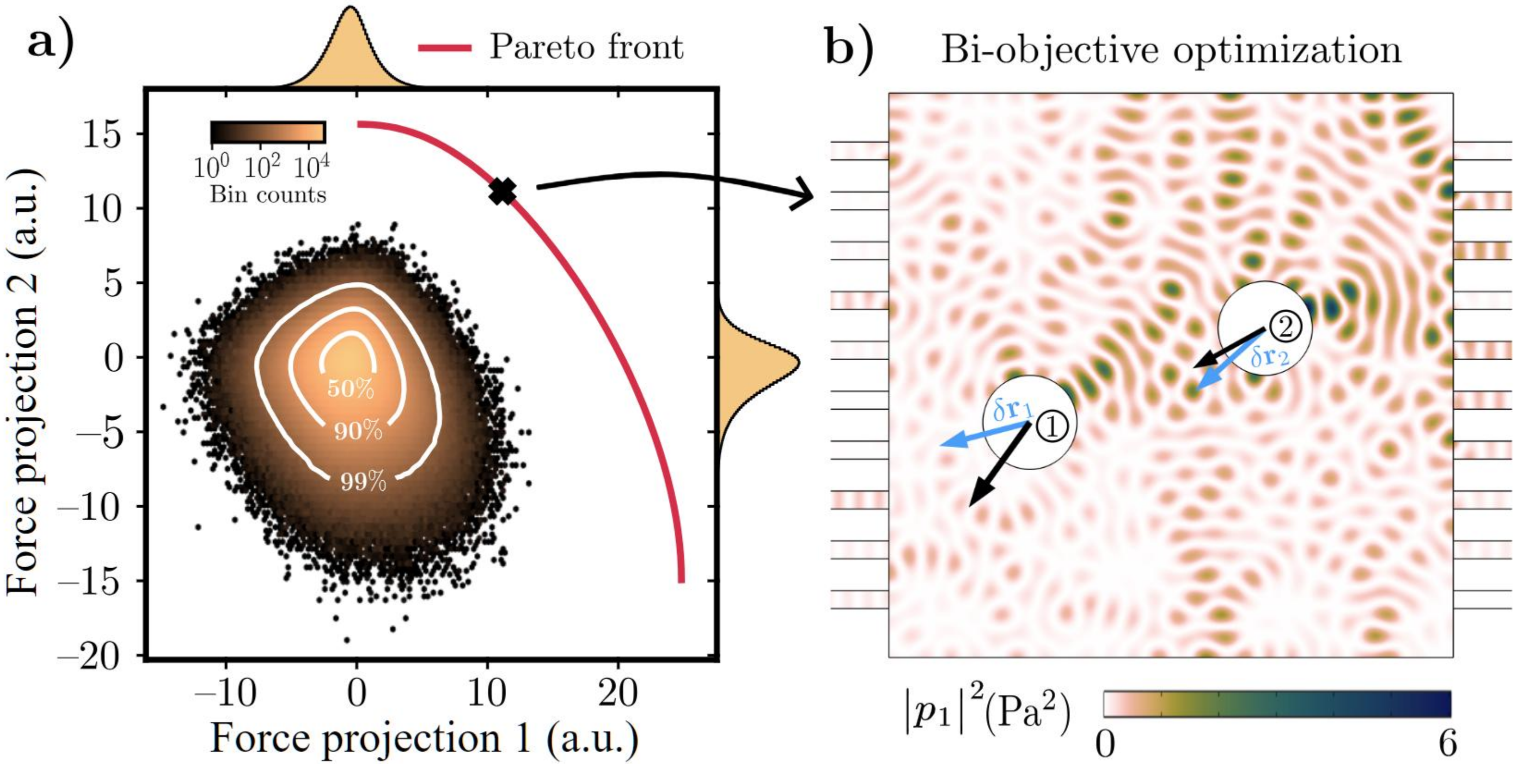
Tristan Nerson





$\|\mathbf{F}_{\text{left}}\| = 26.38$ (a.u.) $\|\mathbf{F}_{\text{right}}\| = 20.90$ (a.u.)

Pareto optimization



How do we find the optima ?

Because Monte-Carlo does not work!

From one object to multiple objects

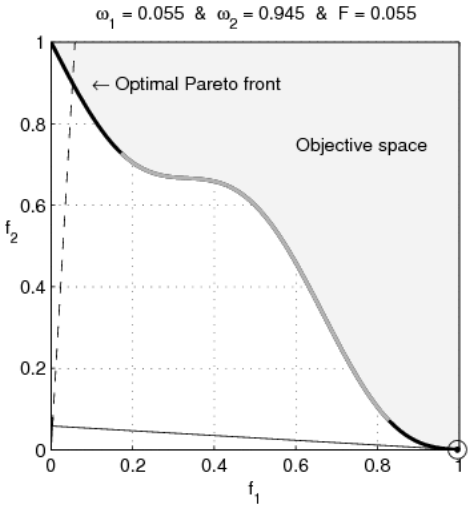
Tristan Nerson

We can write to total virtual work of the radiation force on many objects as:

$$\underbrace{\frac{1}{\omega}\langle \mathbf{Q}_{\delta \mathbf{r}_{\mathrm{tot}}} \rangle}_{\text{Maximizing this maximizes the sum of the projections}} = \frac{1}{|\delta \mathbf{r}_{\mathrm{tot}}|} \sum_{i=1}^{N_d} \underbrace{|\delta \mathbf{r}_i|}_{\text{Optimization weights}} \underbrace{\overline{\mathbf{F}}_i \cdot \hat{\mathbf{e}}_{\mathbf{r}_i}}_{\text{Objectives}}, \qquad \delta \mathbf{r}_{\mathrm{tot}} = (\delta \mathbf{r}_1, \cdots, \delta \mathbf{r}_{N_d})$$

In optimization theory, this is what we call a **linear scalarization**.

For example $\min F(\mathbf{x}) = \omega_1 f_1(\mathbf{x}) + \omega_2 f_2(\mathbf{x})$



Guillaume Jacquenot, NonConvex, Wikimedia Commons, CC BY-SA 3.0

Les Gustins

Linear scalarization only sees the convex hull of the feasible objective set!

From one object to multiple objects

Our feasible objective set is the joint numerical range of Hermitian matrices…

The joint numerical range of a k-tuple of $n \times n$ Hermitian matrices $A = (A_1, \dots, A_k)$ is defined by:

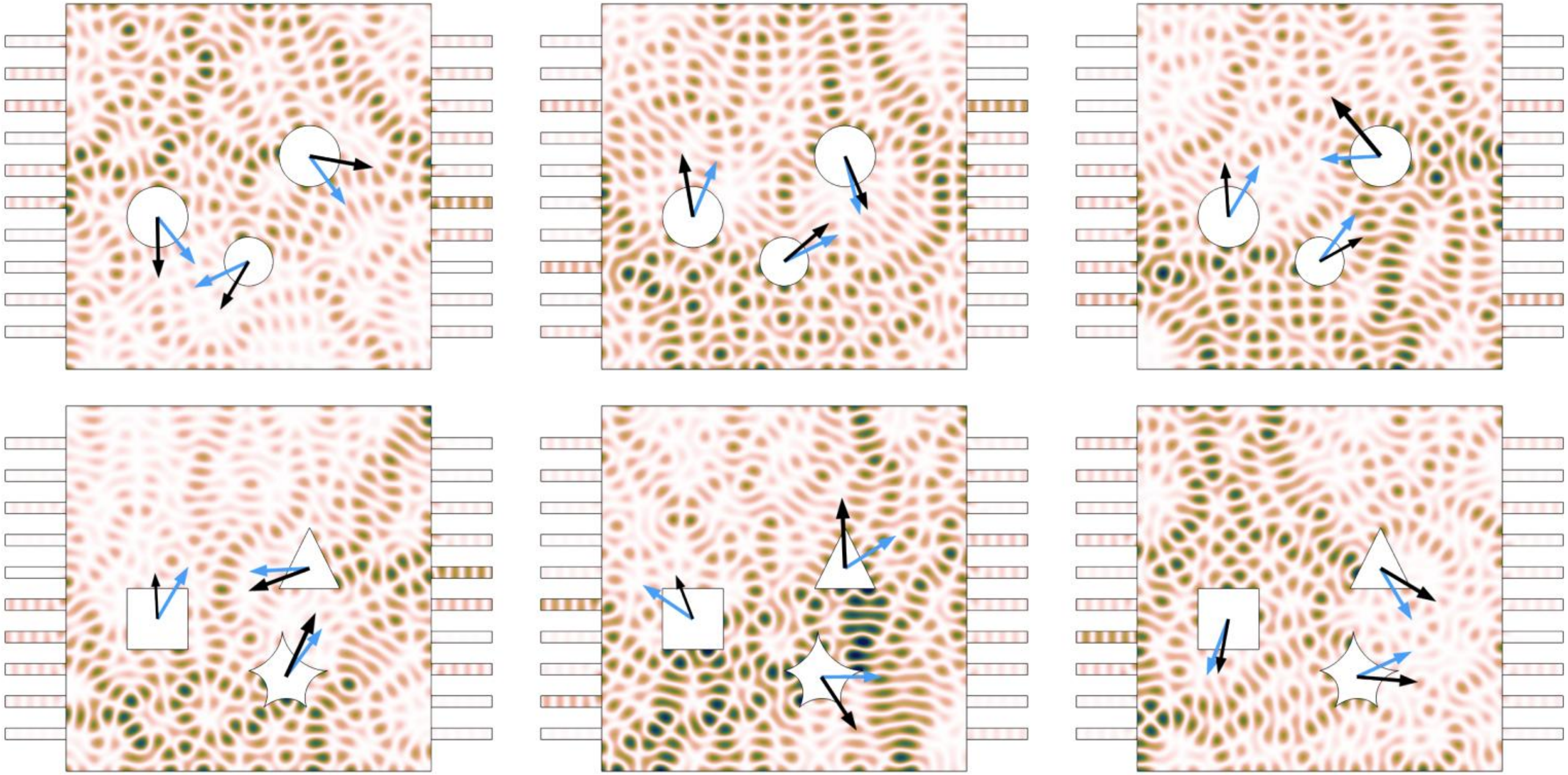
$$W(A_1, \dots, A_k) = \left\{ \left(x^\dagger A_1 x, \dots, x^\dagger A_k x \right) : x \in \mathbb{C}^N, x^\dagger x = 1 \right\}$$

Is this convex?

- The joint numerical range of two Hermitian matrices is always convex (Toeplitz-Hausdorff theorem) → Hausdorff, Felix. Mathematische Zeitschrift 3.1 (1919): 314-316
- The joint numerical range of three Hermitian matrices is convex for matrices of size strictly larger than 2 (Au-Yeung-Poon theorem) → Au-Yeung, Yik Hoi, and Yiu Tung Poon. Southeast Asian Bull. Math 3.2 (1979): 85-92.
- So for strictly more than 3 objectives, we may only retrieve a part of the Pareto front

Side note: objectives vs objects → 3 = 1 + 1 + 1 = 1 + 2

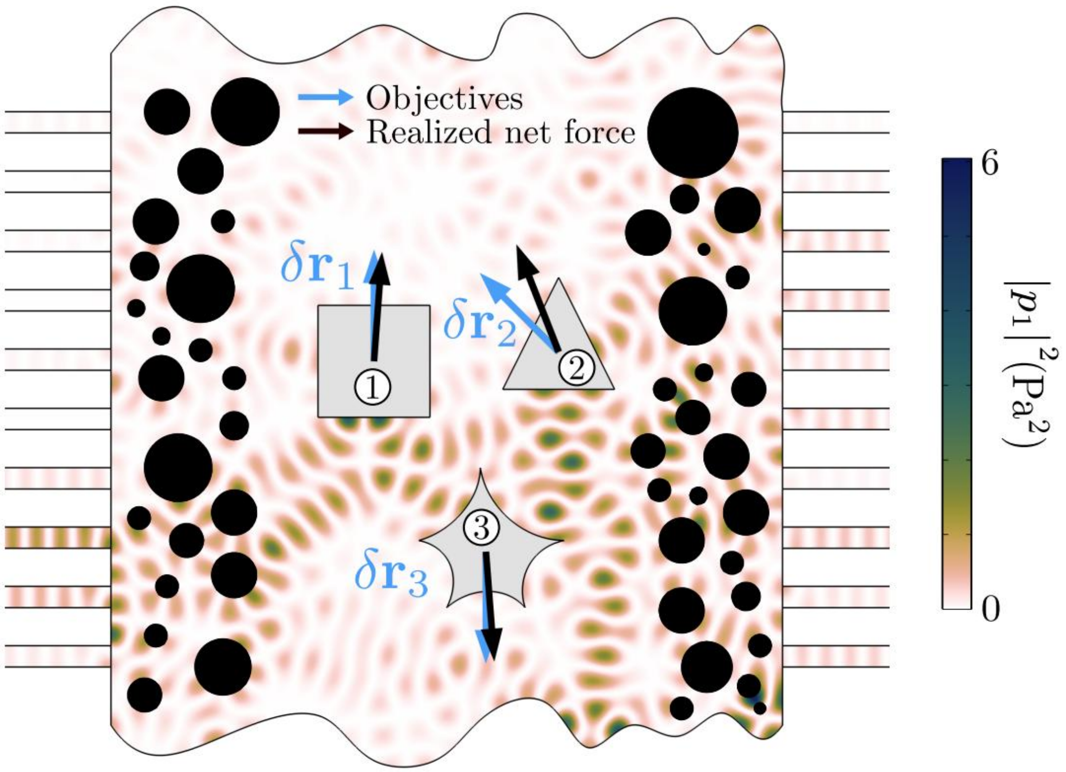
Some examples of tri-objective optimization



Some examples of tri-objective optimization

Tristan Nerson

To sum up, using the GWS matrix is very interesting because it permits to solve a complex multi-objective wave optimization problem by simply writing it as an eigenvalue problem



Simultaneous optimization

- We would like to **simultaneously maximize** the force projections on say 2 objects → Hermitian matrices $\mathbf{Q}_\alpha$ and $\mathbf{Q}_\beta$
- We define the deficits from utopia values (highest eigenvalues of these matrices) as

$$\epsilon_\alpha = \lambda_N(\mathbf{Q}_\alpha) - \langle \mathbf{Q}_\alpha \rangle, \quad \epsilon_\beta = \lambda_N(\mathbf{Q}_\beta) - \langle \mathbf{Q}_\beta \rangle$$

- Using **Schrödinger-Robertson** uncertainty relation + **Bhatia-Davis** inequality:

$$\epsilon_\alpha \epsilon_\beta \geq \frac{\frac{1}{4}\left|\langle [\mathbf{Q}_\alpha, \mathbf{Q}_\beta] \rangle\right|^2 + \mathrm{Cov}(\mathbf{Q}_\alpha, \mathbf{Q}_\beta)^2}{(\langle \mathbf{Q}_\alpha \rangle - \lambda_1(\mathbf{Q}_\alpha))(\langle \mathbf{Q}_\beta \rangle - \lambda_1(\mathbf{Q}_\beta))}$$

$$\mathrm{Cov}(\mathbf{Q}_\alpha, \mathbf{Q}_\beta) = \frac{1}{2}\langle \{\mathbf{Q}_\alpha - \langle \mathbf{Q}_\alpha \rangle, \mathbf{Q}_\beta - \langle \mathbf{Q}_\beta \rangle\} \rangle$$

Simultaneous optimization

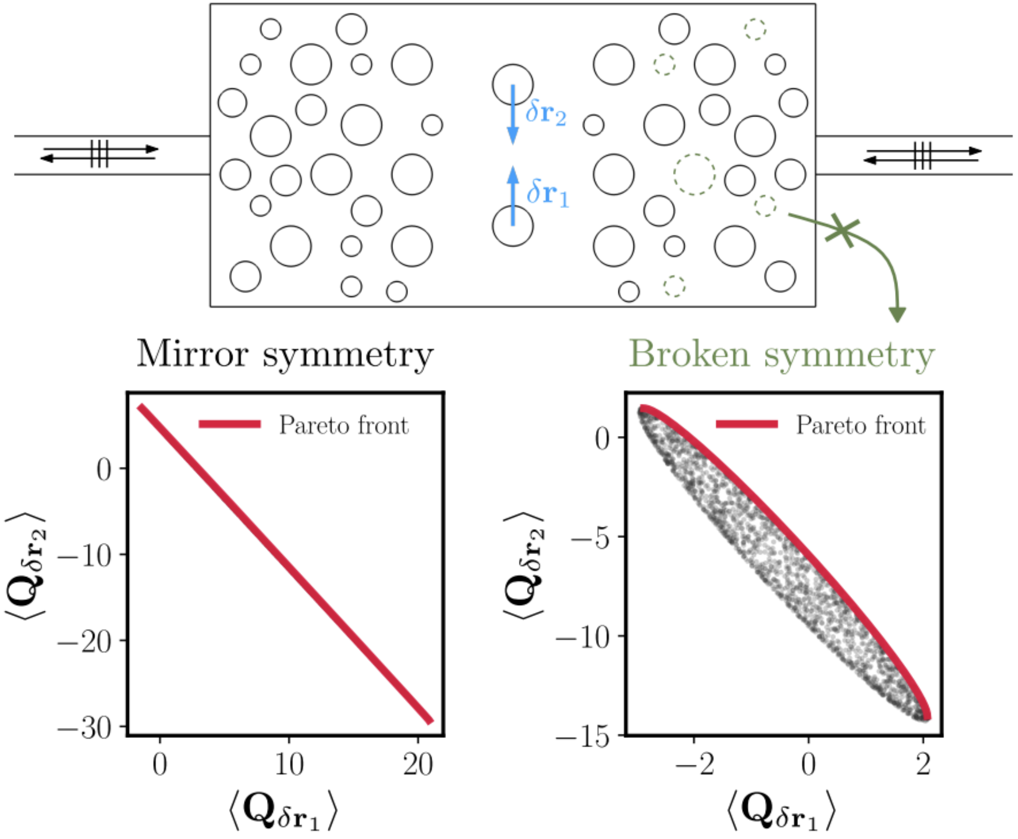
$$\epsilon_\alpha \epsilon_\beta \geq \frac{\frac{1}{4}\left|\langle[\mathbf{Q}_\alpha, \mathbf{Q}_\beta]\rangle\right|^2 + \mathrm{Cov}(\mathbf{Q}_\alpha, \mathbf{Q}_\beta)^2}{(\langle\mathbf{Q}_\alpha\rangle - \lambda_1(\mathbf{Q}_\alpha))(\langle\mathbf{Q}_\beta\rangle - \lambda_1(\mathbf{Q}_\beta))}$$

- RHS can only vanish when the expectation value of the commutator vanish and the two GWS matrices are uncorrelated in the chosen input state
- Ideally, we would like to have $\epsilon_\alpha = \epsilon_\beta = 0$
- Only doable when the two matrices share the same eigenstate associated with their respective maximal eigenvalues

→ **Simultaneous optimization is very non-generic**

When do objectives commute? What happens?

- For example:
  by symmetry!

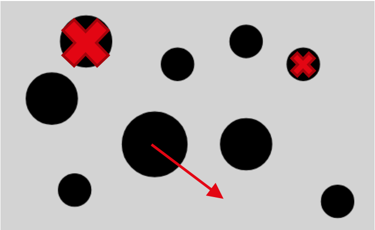


More complicated: constrained optimization

- Now, say that we want to maximize a force/torque projection while ensuring that N_c force projections are 0.
- This is a complex-valued homogeneous Quadratically Constrained Quadratic Program (QCQP):

$$\begin{aligned} \max_{\mathbf{s}_+ \in \mathbb{C}^N} \quad & \langle \mathbf{Q}_\alpha \rangle \\ \text{s.t.} \quad & \langle \mathbf{Q}_{\beta_i} \rangle \triangleright_i b_i, \quad i = 1, \ldots, N_c \\ & \mathbf{s}_+^\dagger \mathbf{s}_+ = 1 \end{aligned}$$

where "$\triangleright$" can signify "$=$", "$\leq$" or "$\geq$"



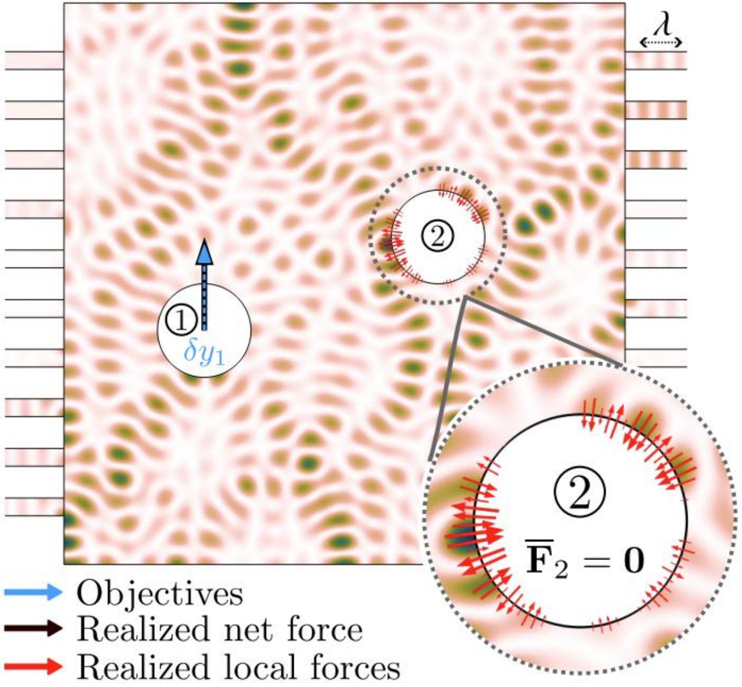
Solving QCQP is generally an NP-hard problem for $N_c > 2$, while we already need 3 constrains in 2D to prescribe the motion of 1 object!

Constrained optimization

GUROBI OPTIMIZATION

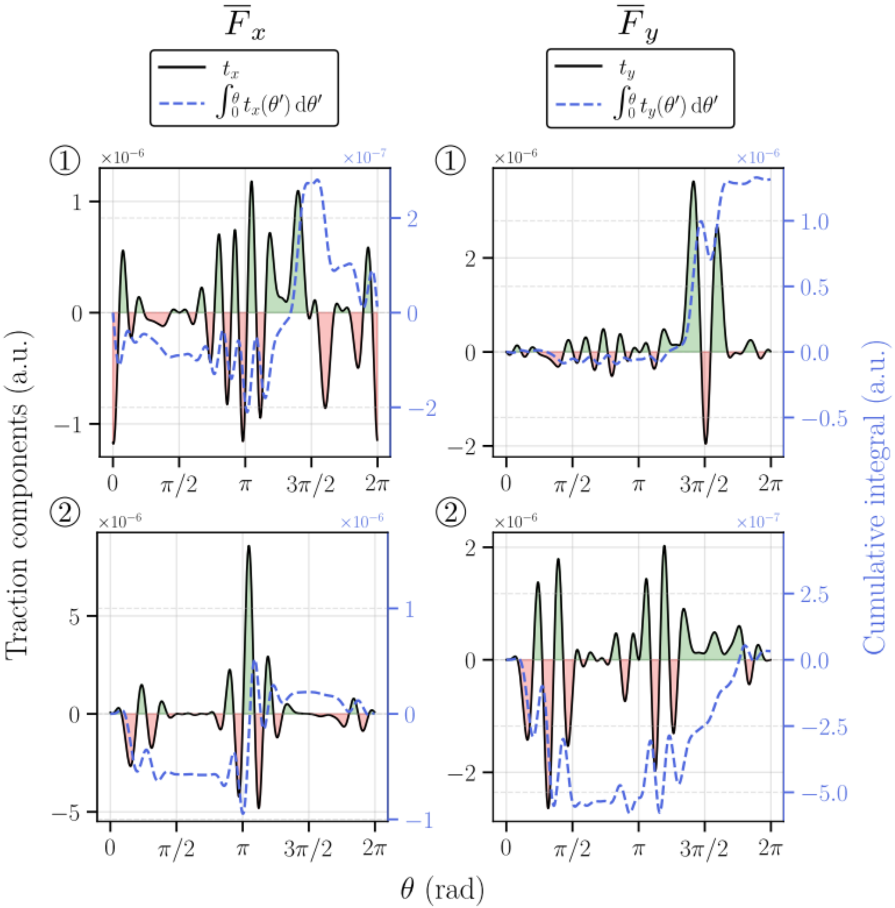
- There are NP-hard solvers, but they take forever to give a global optimum. Local optimum is still *easily* achievable!
- Example: Gurobi solver
  - Max force projection on 1 object, no translation of another object

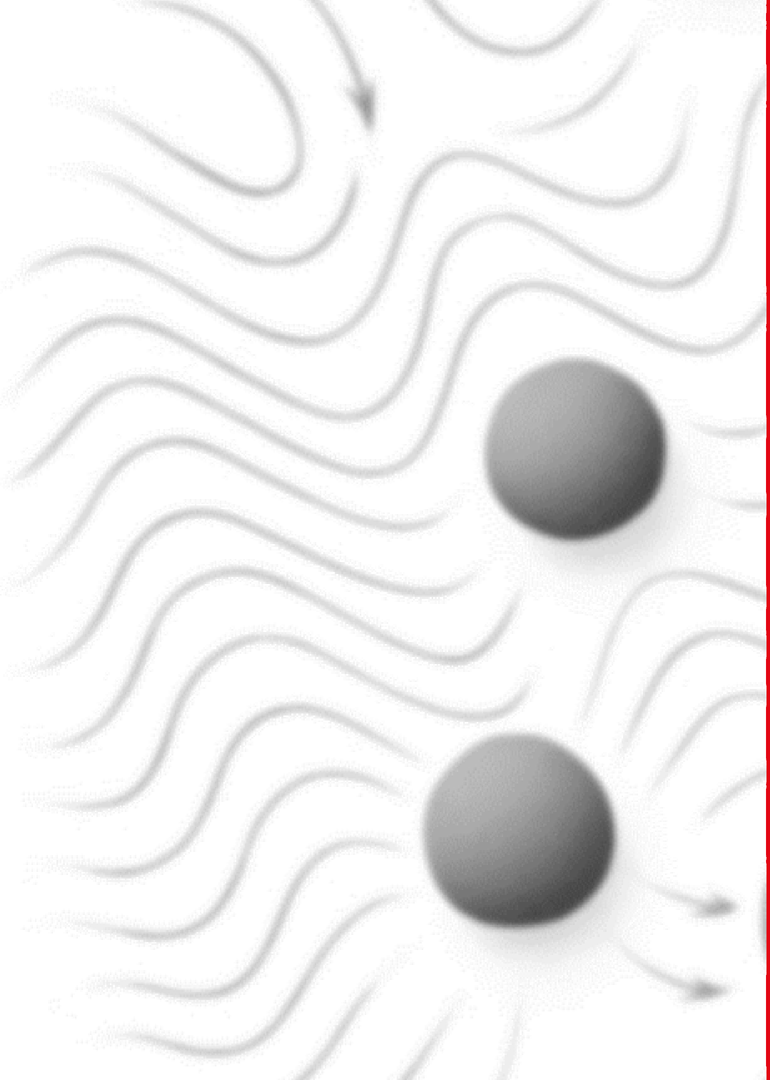
    → Good enough solution in a fraction of second in my local machine



Constrained optimization

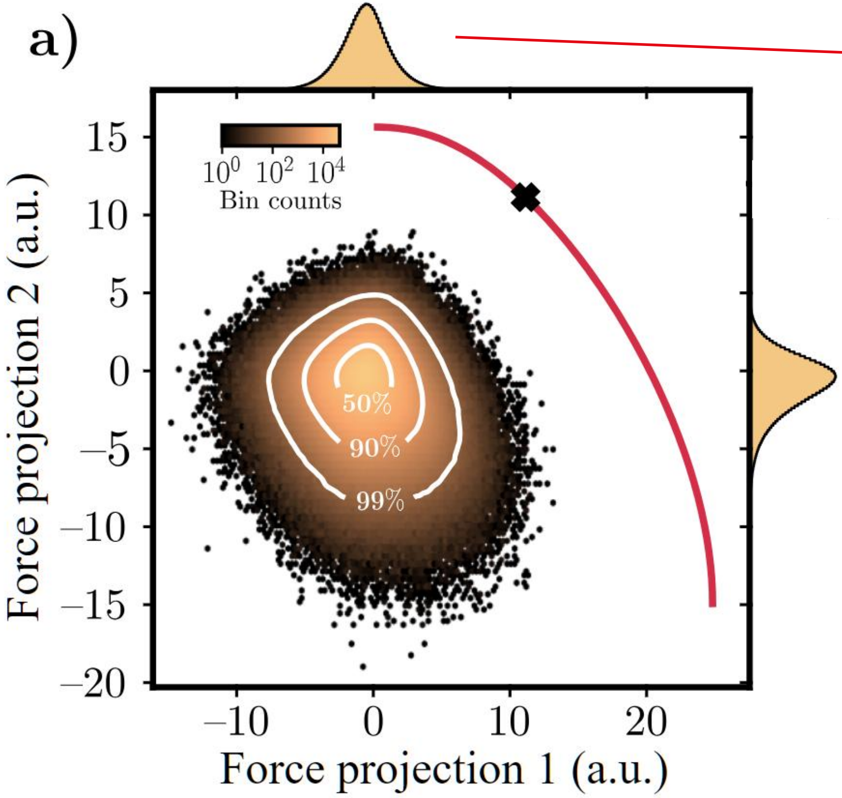
Tristan Nerson





Numerical shadows

Back to the random inputs: Numerical shadows



What are these distribution?

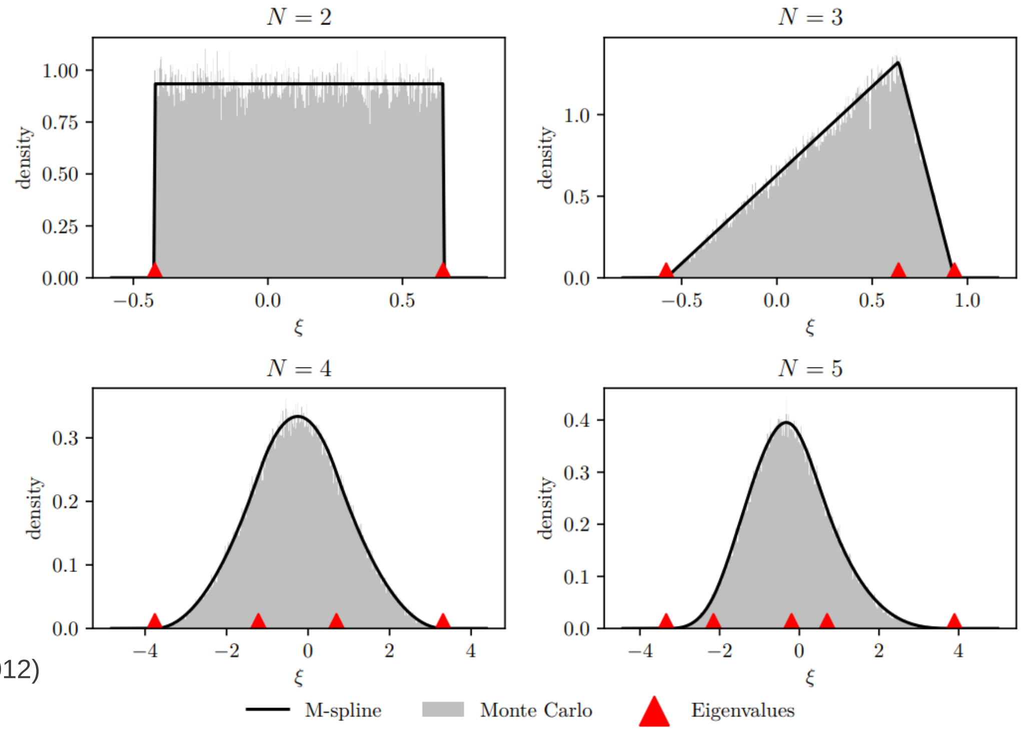
Back to the random inputs: Numerical shadows

The probability density function of the expectation value of a Hermitian matrix is an M-spline (a normalized B-spline with unit integral within the smallest and largest eigenvalues):

$$p(\xi) = (N-1)\sum_{i=1}^{N} \frac{\max(0, \lambda_i - \xi)^{N-2}}{\prod\limits_{j\neq i}(\lambda_i - \lambda_j)}$$

$$\xi \equiv \mathbf{s}_+^\dagger \mathbf{Q}_\alpha \mathbf{s}$$

Example: random Hermitian matrices



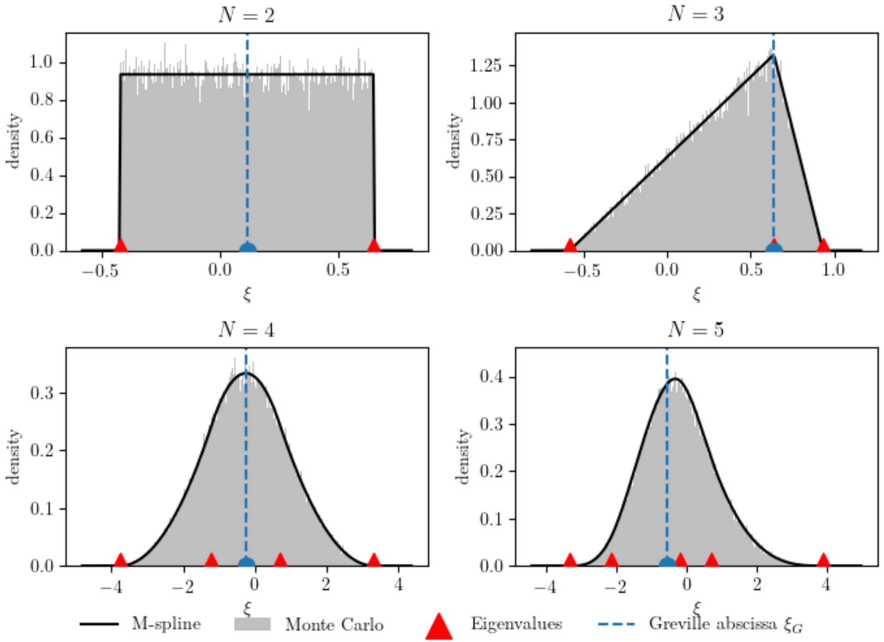
Gallay and Serre. Commun. Pure App. Math. 65, 287 (2012)
Dunkl, et al., Linera Alebra Appl. 434, 2042 (2011)
Venuti and Zanardi, Phys. Lett. A 277, 1854 (2013)

Back to the random inputs: Numerical shadows

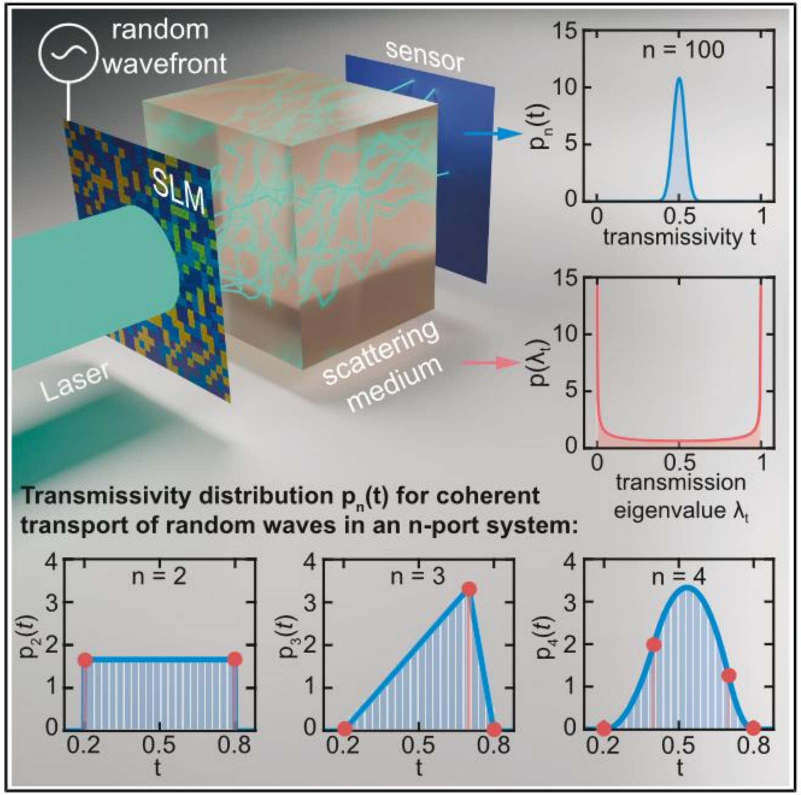
Where is the maximum (mode of the distribution)? → close to the Greville abscissa

Wang and Guo, Newton 2, 100478 (2026)

$$\xi_{\mathrm{mode}} = \arg\max_{\xi} p(\xi) \approx \frac{1}{N-2}\sum_{i=2}^{N-1} \lambda_i$$



Numerical shadows: transmission & reflection



Wang and Guo, Newton 2, 100478 (2026)

$$\underset{[(m_L+m_R)\times(m_L+m_R)]}{\mathbf{S}} = \begin{pmatrix} \underset{[m_L\times m_L]}{\mathbf{r_L}} & \underset{[m_L\times m_R]}{\mathbf{t_{R\to L}}} \\ \underset{[m_R\times m_L]}{\mathbf{t_{L\to R}}} & \underset{[m_R\times m_R]}{\mathbf{r_R}} \end{pmatrix}$$

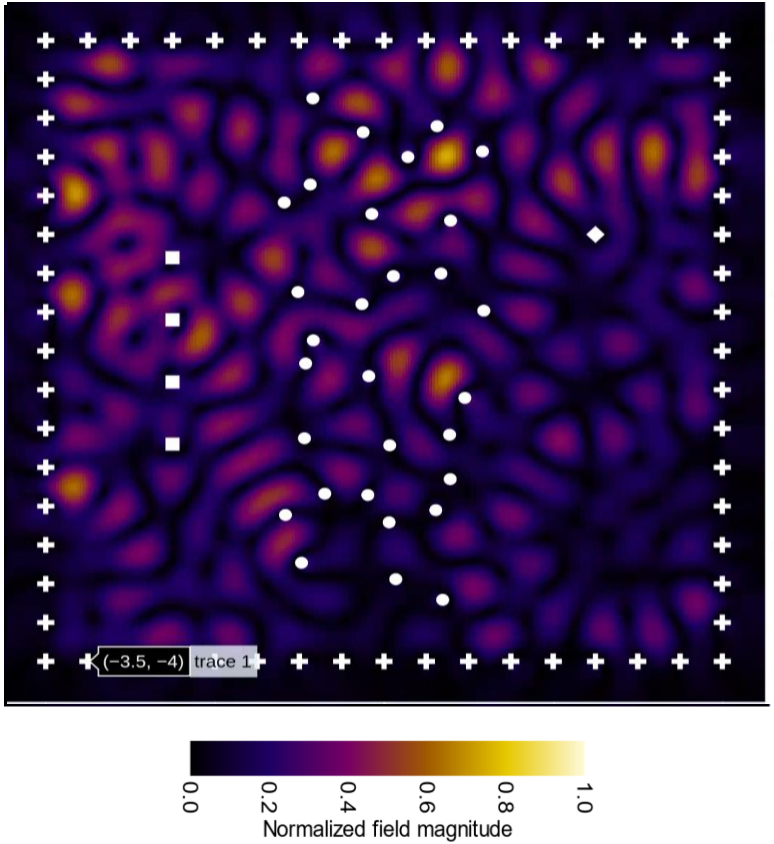
- **Transmision → transmittance matrix**

$$P_{\text{trans}} = \langle \mathbf{t}^\dagger \mathbf{t} \rangle_{\mathbf{s}_+}$$

- **Reflection → reflectance matrix**

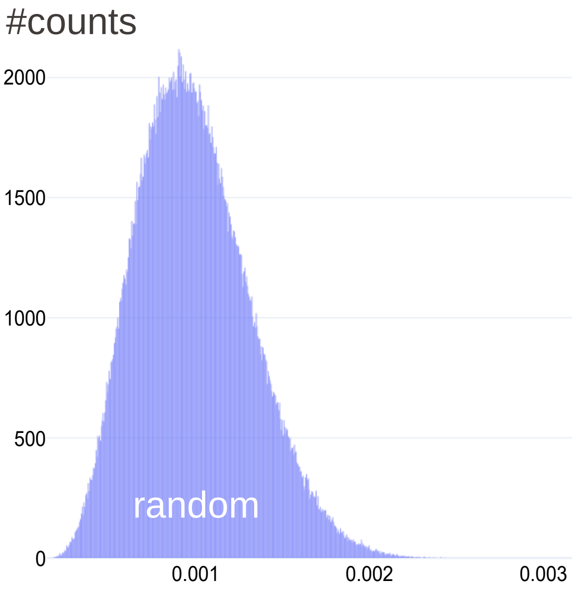
$$P_{\text{refl}} = \langle \mathbf{r}^\dagger \mathbf{r} \rangle_{\mathbf{s}_+}$$

Numerical shadows: focusing in coupled dipole theory



$$\max_{\mathbf{E}\in\mathbb{C}^N} \quad \langle \mathbf{C} \rangle_{\mathbf{E}}$$
$$\text{s.t.} \quad \mathbf{E}^\dagger \mathbf{E} = 1$$

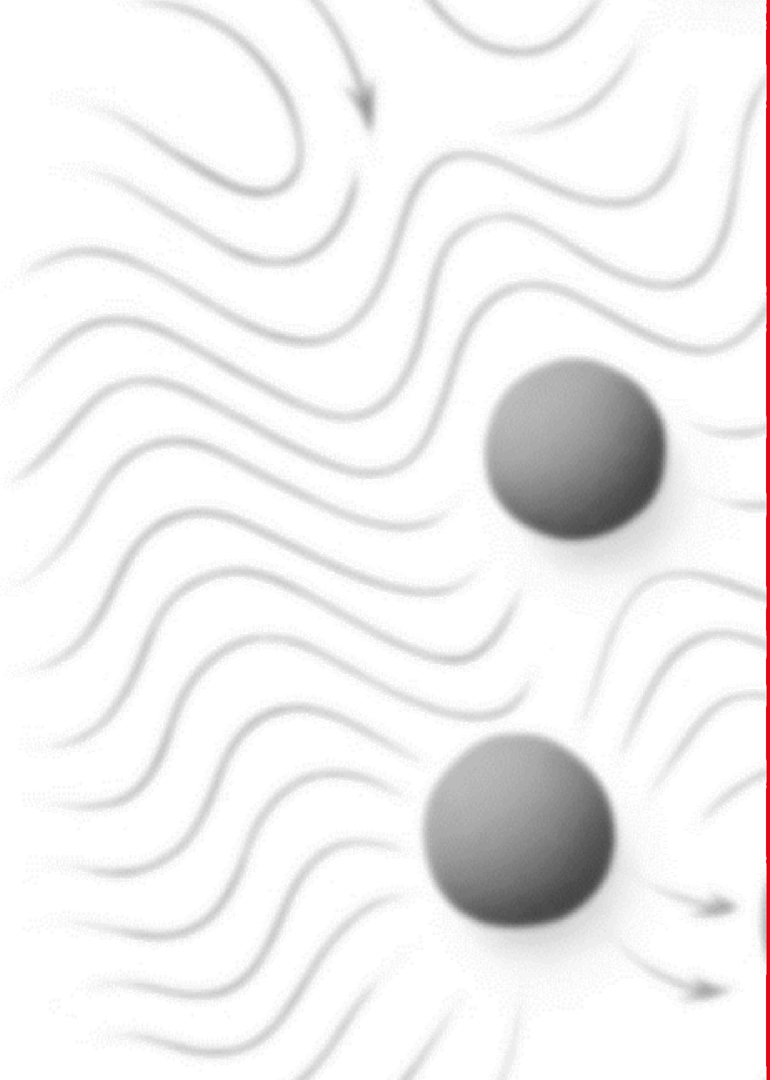
Coupling matrix telling us how efficiently an incident field pattern couples to the selected output



Courtesy of T. Tuuva and R. Fleury. Paper in prep.

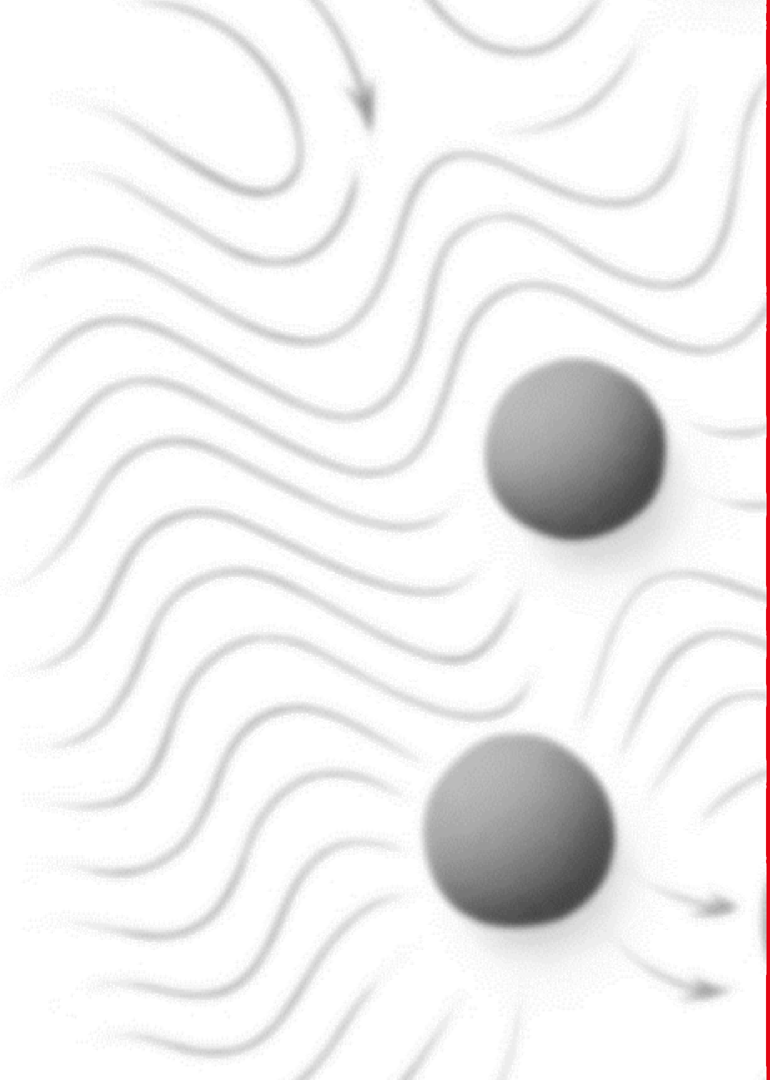
Les Gustins

Tristan Nerson



BREAK

Tristan Nerson



Rewriting the problem with other degrees of freedom

Voltage and current formalism

$$\mathbf{i} = \mathbf{Y}\mathbf{u}$$

$$\mathbf{i} = \sqrt{2}(\mathbb{1} - \mathbf{S})\mathbf{s}_+$$

Current

$$\mathbf{Y} = \mathbf{G} + j\mathbf{B}$$

Admittance matrix

Conductance

Susceptance

$$\mathbf{u} = \sqrt{2}(\mathbb{1} + \mathbf{S})\mathbf{s}_+$$

Voltage

$$P_{\mathrm{abs}} = \frac{1}{2}\mathrm{Re}[\mathbf{u}^\dagger\mathbf{i}] = \mathbf{s}_+^\dagger\mathbf{s}_+ - \mathbf{s}_-^\dagger\mathbf{s}_- = \frac{1}{2}\mathbf{u}^\dagger\mathbf{G}\mathbf{u} = \frac{1}{2}\langle\mathbf{G}\rangle_{\mathbf{u}}$$

$$P_{\mathrm{rea}} = -\frac{1}{2}\mathrm{Im}[\mathbf{u}^\dagger\mathbf{i}] = 2\,\mathrm{Im}[\mathbf{s}_+^\dagger\mathbf{s}_-] = -\frac{1}{2}\mathbf{u}^\dagger\mathbf{B}\mathbf{u} = -\frac{1}{2}\langle\mathbf{B}\rangle_{\mathbf{u}}$$

Zavatski, Nerson, Fleury and Martin. arXiv preprint arXiv:2605.31151 (2026)

Voltage and current formalism

For lossless systems: $\mathbf{S}^\dagger \mathbf{S} = \mathbb{1} \qquad \mathbf{Y} = j\mathbf{B}$

We can then show that: $\langle \mathbf{Q}_{\delta \mathbf{r}} \rangle_{\mathbf{s}_+} = \frac{1}{4} \langle \delta \mathbf{r} \cdot \nabla_{\delta \mathbf{r}} \mathbf{B} \rangle_{\mathbf{u}}$

Finally: $\boxed{\mathbf{F} \cdot \delta \mathbf{r} = -\frac{1}{4\omega} \langle \delta \mathbf{r} \cdot \nabla_{\delta \mathbf{r}} \mathbf{B} \rangle_{\mathbf{u}}}$

We rewrote the problem in terms of a **voltage quadratic form**

→ **We do optimization on the voltage**

→ **We optimize at fixed $\mathbf{u}^\dagger \mathbf{u}$**

Waves versus quasistatic regimes

Are the controlled and measured DoF propagating channels?

Waves

Typically, $kL \simeq 1$ or $kL \gg 1$

Propagation phase across the system is important → motivation to distinguish *forward* and *backward* waves

$$\mathbf{s}_{-} = \mathbf{S}\mathbf{s}_{+}$$

Spatial Light Modulator, intensity on a camera, fiber-mode amplitudes, microwave Vector Network Analyzer, loudspeaker/microphones arrays, high frequency transducers…

Quasistatic

Typically, $kL \ll 1$

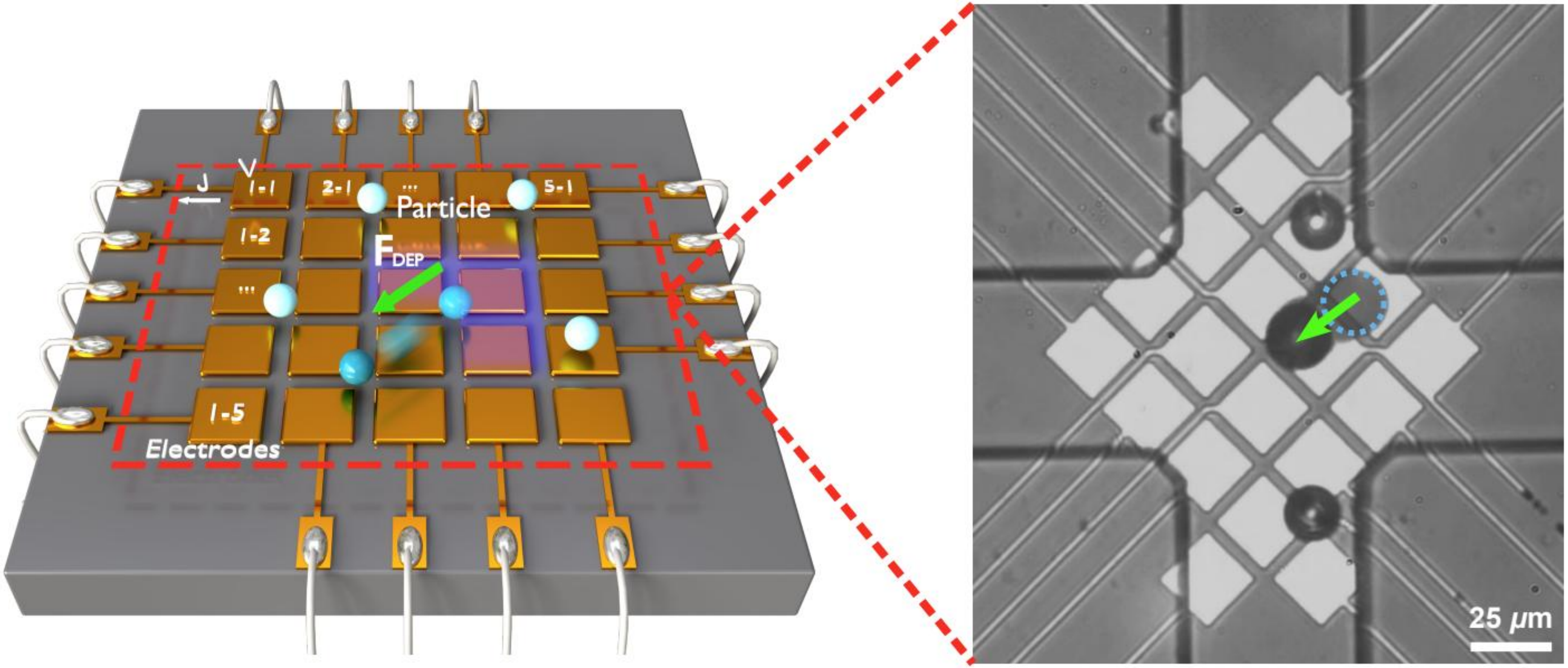
Propagation phase across the system is negligible → no motivation to distinguish *forward* and *backward* waves

$$\mathbf{i} = \mathbf{Y}\mathbf{u}$$

Electrode voltage and current, capacitive touchscreen, electrical impedance tomography,…

Particle manipulation in lossy systems:

the low frequency electromagnetic case



Zavatski, Nerson, Fleury and Martin. arXiv preprint arXiv:2605.31151 (2026)

Equivalence principle

In electroquasistatic regime, the electric field is irrotational and satisfies:

$$\nabla \cdot [(\sigma(\mathbf{r}) + j\omega\varepsilon(\mathbf{r}))\nabla\phi(\mathbf{r})] = 0, \qquad \mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

$$= -\nabla \cdot \mathbf{J}_{\text{tot}}$$

Low frequency lossy problem: (wavelength of approx. 60m, for objects that are few microns)

$$\mathbf{Y} = \mathbf{G} \qquad \omega\varepsilon(\mathbf{r}) \ll \sigma(\mathbf{r}), \quad \nabla \cdot (\sigma(\mathbf{r})\nabla\phi(\mathbf{r})) = 0$$

Auxiliary lossless problem. Imagine a system with: $\sigma^\sharp = 0, \quad \varepsilon^\sharp(\mathbf{r}) = \dfrac{\sigma(\mathbf{r})}{\omega}$

$$\nabla \cdot (j\omega\varepsilon^\sharp(\mathbf{r})\nabla\phi^\sharp(\mathbf{r})) = 0 \implies \nabla \cdot (\sigma(\mathbf{r})\nabla\phi^\sharp(\mathbf{r})) = 0$$

With the same electrode voltages, the auxiliary problem is the same BVP as the original one.

$$\phi^\sharp = \phi, \quad \mathbf{E}^\sharp = \mathbf{E}$$

Equivalence principle

Ohm's law (free current density): $\mathbf{J} = \sigma \mathbf{E}$

Displacement current density: $\mathbf{J}^\sharp = j\omega\varepsilon^\sharp \mathbf{E}^\sharp = j\mathbf{J}$

After integration on the electrodes: $\mathbf{i}^\sharp = j\mathbf{i} = j\mathbf{G}\mathbf{u} \equiv \mathbf{Y}^\sharp \mathbf{u}$

We obtain: $\mathbf{Y}^\sharp = j\mathbf{G} \equiv j\mathbf{B}^\sharp$

Then: $$\mathbf{F}^\sharp \cdot \delta\mathbf{r} = -\frac{1}{4\omega}\langle \delta\mathbf{r} \cdot \nabla_{\delta\mathbf{r}} \mathbf{B}^\sharp \rangle_{\mathbf{u}}$$

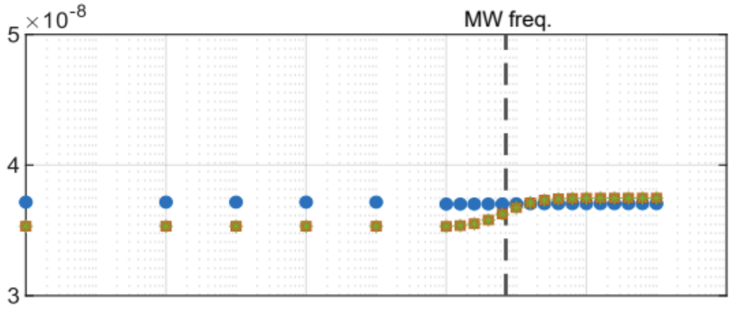
How does this relate to the usual force?

We can show* that: $$\mathbf{F} = \frac{\varepsilon_{\text{liq}}}{\varepsilon^\sharp_{\text{liq}}}\mathbf{F}^\sharp = \frac{\varepsilon_{\text{liq}}\omega}{\sigma_{\text{liq}}}\mathbf{F}^\sharp \rightarrow \boxed{\mathbf{F} \cdot \delta\mathbf{r} = -\frac{\varepsilon_{\text{liq}}}{4\sigma_{\text{liq}}}\langle \delta\mathbf{r} \cdot \nabla_{\delta\mathbf{r}} \mathbf{G} \rangle_{\mathbf{u}}}$$

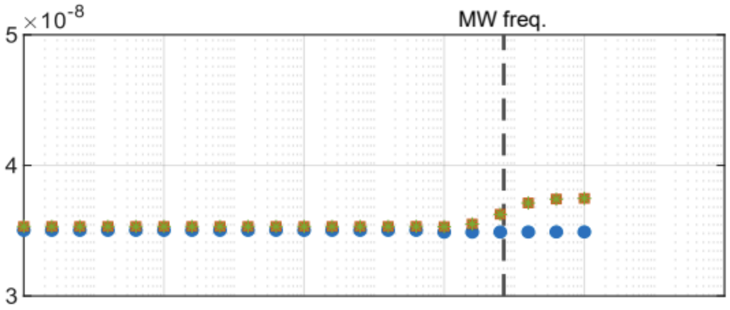
*Full derivation with the Minkowksi stress tensor: Zavatski, Nerson, Fleury and Martin. arXiv preprint arXiv:2605.31151 (2026)

Equivalence principle

B-matrix



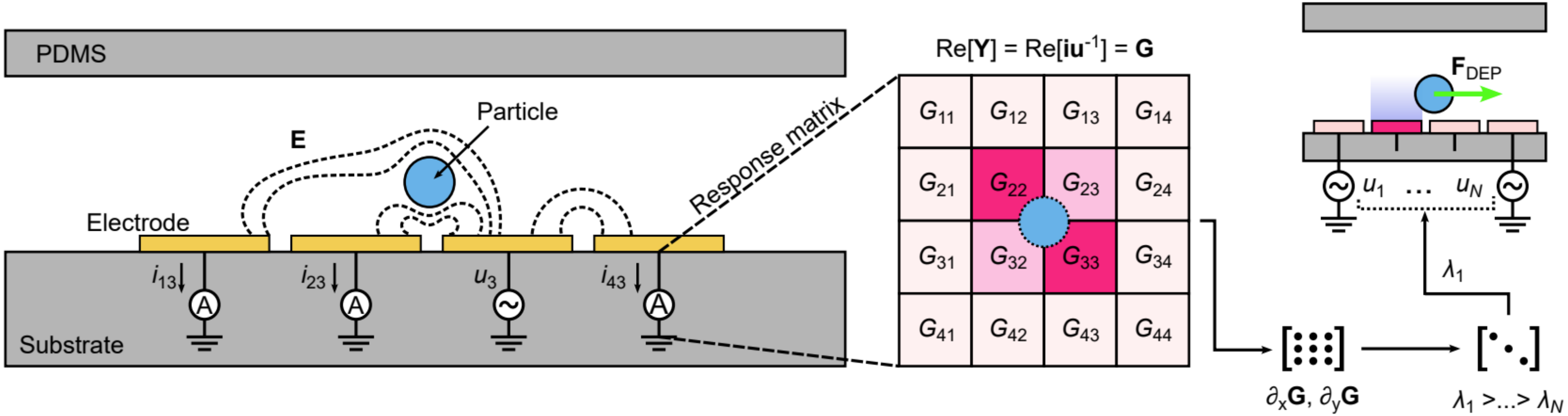
G-matrix



Particle manipulation in lossy systems:

the low frequency electromagnetic case

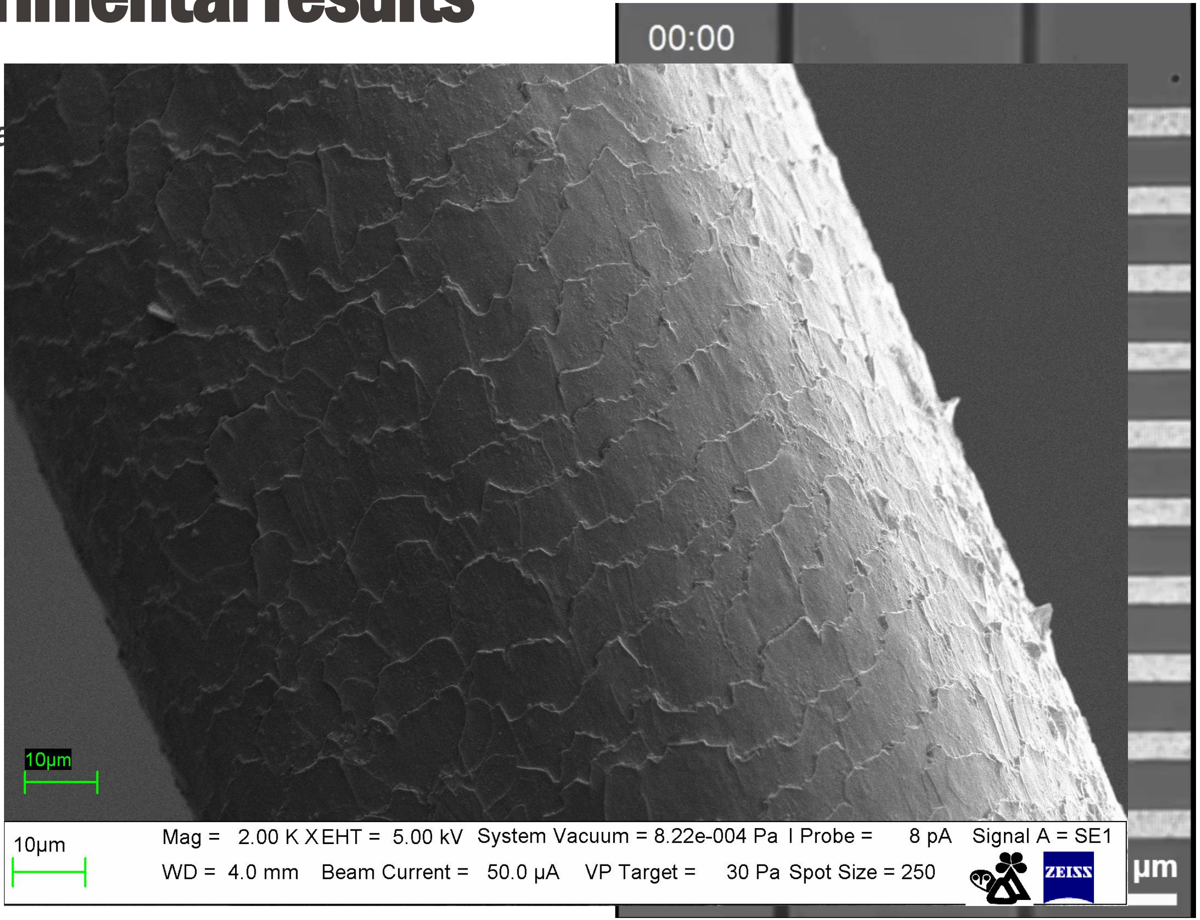
Tristan Nerson



Zavatski, Nerson, Fleury and Martin. arXiv preprint arXiv:2605.31151 (2026)

Experimental results

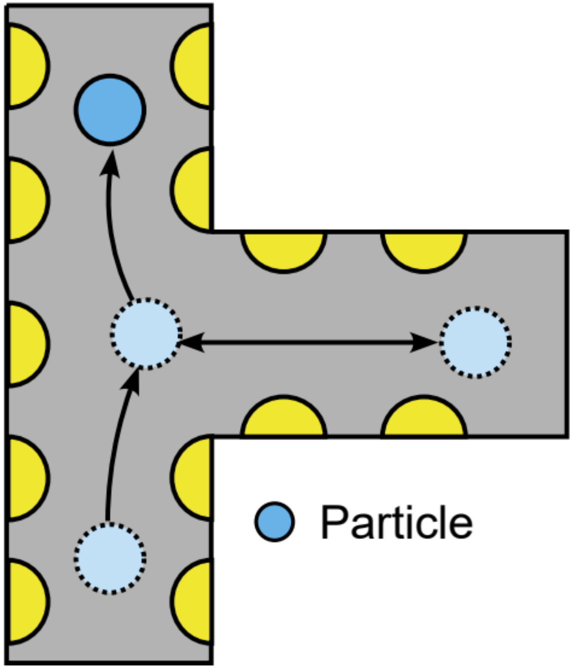
Linea

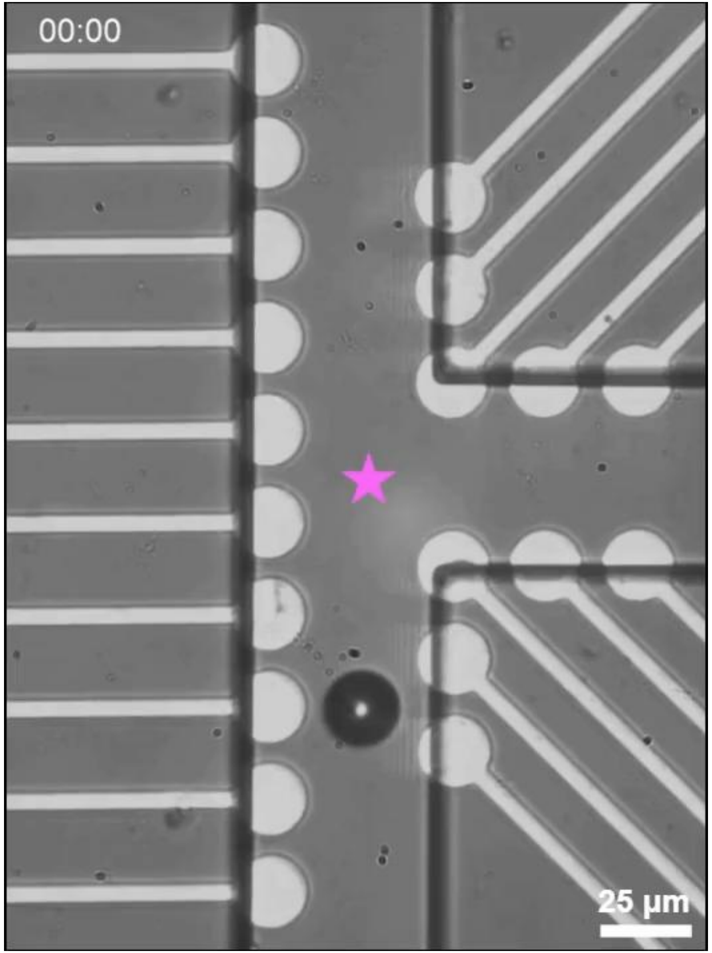


Nicola Angeli/MUSE, Human Hair 2000X, Wikimedia Commons, CC BY-SA 3.0

Experimental results

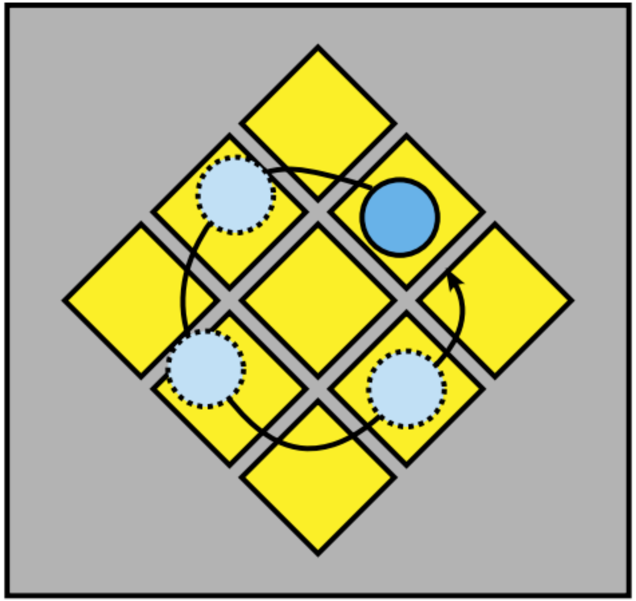
1D motion

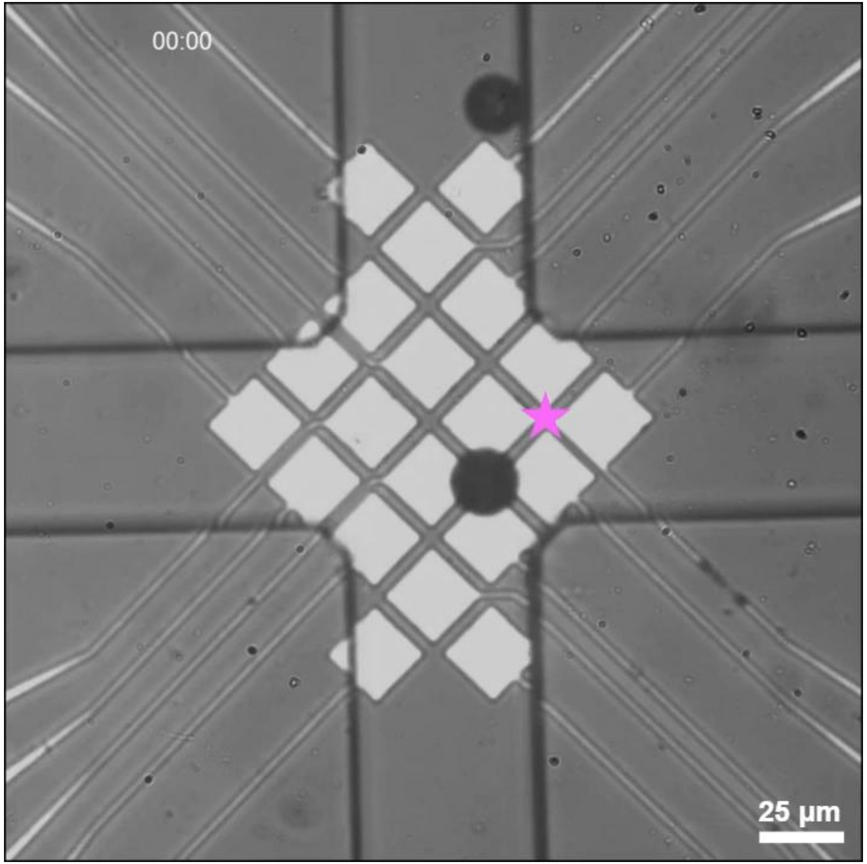




Experimental results

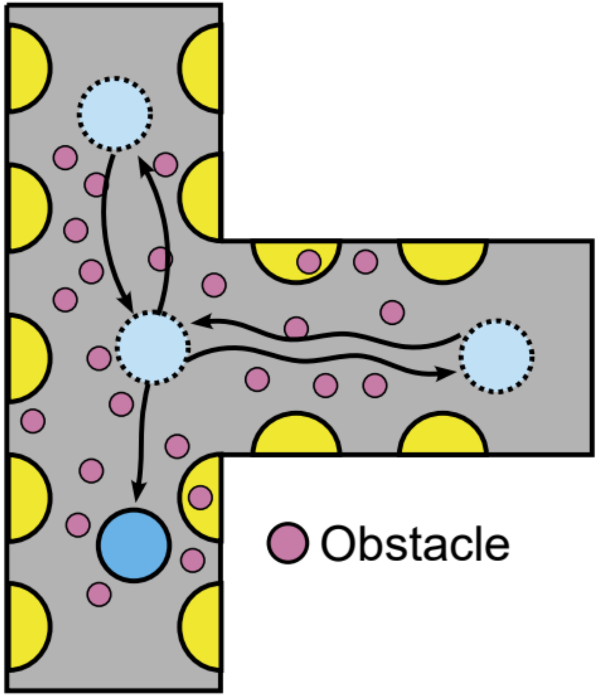
2D motion

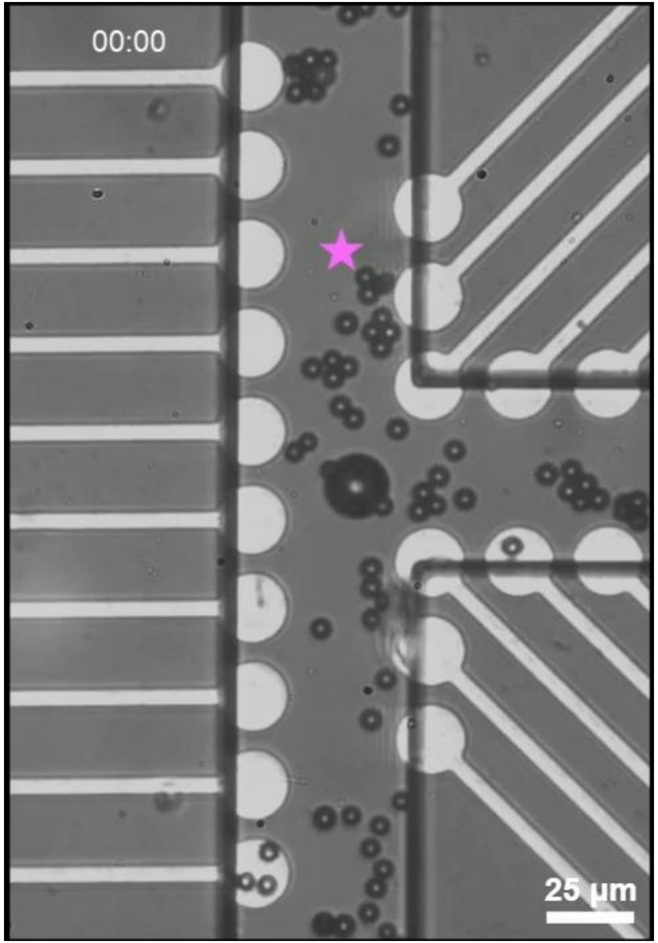


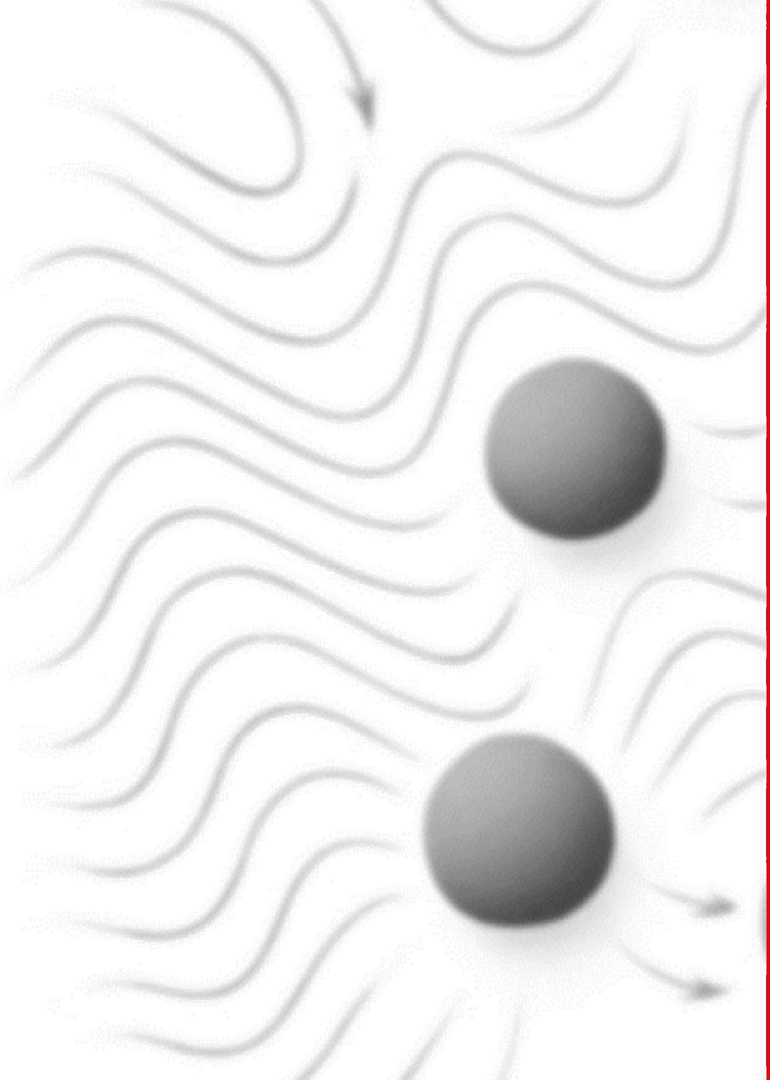


Experimental results

Disordered background







Conclusion

Conclusion

The physics constructs the operator; linear algebra selects the wavefront.

- Single objective: $$\max_{\mathbf{x}^\dagger \mathbf{x}=1} \mathbf{x}^\dagger \mathbf{A} \mathbf{x} = \lambda_{\max}(\mathbf{A})$$

- Several objectives: weighted sums of Hermitian operators generate Pareto-optimal wavefronts; additional constraints generally produce nonconvex QCQPs

- The same principles survive when scattering amplitudes are replaced by voltages and currents. In some regimes, this conversion is very useful!

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Thank you!

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